Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Geometry and Physics 56 (2006) 2345–2378
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Orbifold projective structures, differential operators, and logarithmic connections on a pointed Riemann surface

Indranil Biswas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

Received 13 October 2005; received in revised form 21 December 2005; accepted 21 December 2005
Available online 30 January 2006

Abstract

Defining orbifold projective structures on a multi-pointed compact Riemann surface, we give a necessary and sufficient condition for the existence of such a structure. Orbifold projective structures are described using logarithmic connections, as well as using third order holomorphic differential operators.

MSC: 32C38; 14F10; 14H60

Keywords: Orbifold projective structure; Differential operator; Connection

1. Introduction

A projective structure on a Riemann surface is defined by giving a holomorphic coordinate atlas such that all the transition functions are Möbius transformations. After fixing a theta characteristic on a compact Riemann surface X, a projective structure gives a second order holomorphic differential operator on X, which has the property that the monodromy of the local system, defined by the sheaf of solutions of the differential operator, is in $\mathrm{SL}(2, \mathbb{C})$. This way, projective structures correspond to flat $\mathrm{SL}(2, \mathbb{C})$-bundles with a line subbundle whose second fundamental form is an isomorphism (see [9]).

If E is the flat vector bundle of rank two over X corresponding to a projective structure on X, then the adjoint bundle $\mathrm{ad}(E)$ is holomorphically identified with the second order jet bundle $J^2(TX)$, where TX is the holomorphic tangent bundle of X (Proposition 4.1). This way, projective structures on X get identified with all flat (holomorphic) connections on $J^2(TX)$

E-mail address: indranil@math.tifr.res.in.

doi:10.1016/j.geomphys.2005.12.005

satisfying certain compatibility conditions with the Lie bracket operation of vector fields (see Theorem 4.5 for the details). The local systems on X corresponding to the flat connections on $J^2(TX)$ that arise from projective structures on X are identified with the local systems given by the solutions of a certain class of third order holomorphic differential operators from TX to $(T^*X)^{\otimes 2}$ (see Section 6.1).

The aim here is to systematically investigate the orbifold analog of projective structures on a compact Riemann surface.

Let X be a compact connected Riemann surface and $\mathbb{D} \subset X$ a finite subset. For each point $\zeta \in \mathbb{D}$, fix an integer $\varpi(\zeta) \geq 2$. Fixing such a data, we define an orbifold projective structure on X to be a covering of X by ramified covering coordinates, that is, ramified holomorphic maps from open subsets of $\mathbb{C}$ to open subsets of X ramified only over $\mathbb{D}$ with the indices of ramification governed by the function ϖ, such that all the local transition functions arise from Möbius transformations (see Section 3.2 for the details).

In Lemma 3.2 we show that X admits an orbifold projective structure if and only if at least one of the following three conditions holds:

(1) $\text{genus}(X) \geq 1$;
(2) $\#\mathbb{D} \neq 1, 2$;
(3) $\#\mathbb{D} = 2$ and ϖ is a constant function.

In other words, X does not admit any orbifold projective structure if and only if all the following three conditions hold:

(1) $\text{genus}(X) = 0$,
(2) $\#\mathbb{D} \in \{1, 2\}$, and
(3) if $\mathbb{D} = \{\zeta_1, \zeta_2\}$, then $\varpi(\zeta_1) \neq \varpi(\zeta_2)$.

A key input in the proof of Lemma 3.2 is a theorem of Bundgaard-Nielsen and Fox.

Since the line bundle $TX \bigotimes \mathcal{O}_X(\mathbb{D})$ over X need not admit a square-root (when $\#\mathbb{D}$ is odd it does not have a square-root), orbifold projective structures cannot, in general, be described by second order differential operators between some holomorphic line bundles.

We characterize orbifold projective structures on X in terms of third order singular holomorphic differential operators on X (Theorem 6.1). Orbifold projective structures are also characterized in terms of logarithmic connections on X singular over $\mathbb{D}$ (Theorem 5.2).

In [11], ramified projective structures on X were defined using ramified coordinate maps *from* open subsets of X, while here we define orbifold projective structures using ramified maps *to* X. Note that given any X, if the number of ramification points is sufficiently large, then there are no ramified projective structures on X (see [11, page 267, Theorem 3]).

When $\mathbb{D} = \emptyset$, some of the results proved here were obtained in [7]. The present work was also inspired by [3]. See [2] for generalizations of projective structures.

In [10], the uniformization of a compact Riemann surface was investigated using Higgs bundles (see [10, Section 11]). In [6] (also in [13]), a similar study was carried out for orbifold Riemann surfaces. A projective structure on a Riemann surface X of genus at least two gives an irreducible flat connection of rank two on X. Therefore, a projective structure on X gives Higgs bundle over X of rank two (see [10]). It would be interesting to identify all the Higgs bundles over X that arise this way.

2. Jet bundles and differential operator

2.1. Jet bundle

Let X be a compact connected Riemann surface. The self-product $X \times X$ will be denoted by Z. Let

$$\Delta \subset Z$$

be the (reduced) diagonal divisor in the complex surface Z consisting of all points of the form (x, x). Let

$$p_i : Z \longrightarrow X,$$

$i = 1, 2$, denote the projection of $X \times X$ to the i-th factor of the Cartesian product.

Notation. For a complex manifold Y, the sheaf of holomorphic functions on it will be denoted by $\mathcal{O}_Y$, and for a divisor D on Y, the holomorphic line bundle over Y defined by D will be denoted by $\mathcal{O}_Y(D)$.

Since Δ is an effective divisor on Z, for any holomorphic vector bundle V over Z and any integer $i \geq 1$, the coherent sheaf defined by the holomorphic sections of V is naturally a subsheaf of the coherent sheaf defined by the sections of $V \bigotimes_{\mathcal{O}_Z} \mathcal{O}_Z(i\Delta)$.

Let E be a holomorphic vector bundle over X. For any integer $k \geq 0$, the k-th order *jet bundle* of E, denoted by $J^k(E)$, is defined to be the following direct image on X :

$$J^k(E) := p_{1*}\left(\frac{p_2^* E}{p_2^* E \otimes \mathcal{O}_{X\times X}(-(k+1)\Delta)}\right).$$

So $J^k(E)$ is a holomorphic vector bundle of rank $(k + 1) \cdot \operatorname{rank}(E)$ over X.

Let K_X denote the holomorphic cotangent bundle of X. For any $k \geq 0$, let

$$f_{\mathcal{O},k} : K_X^{\otimes k} \longrightarrow J^k(\mathcal{O}_X)$$

be the homomorphism defined as follows. Take a point $x \in X$ and a holomorphic function f defined on some analytic open subset of X containing x with $f(x) = 0$. The homomorphism $f_{\mathcal{O},k}(x)$ sends the tensor power $(\mathrm{d}f)^{\otimes k}(x) \in (K_X^{\otimes k})_x$ to the element in $J^k(\mathcal{O}_X)_x$ defined by the function $(f)^k/k!$. To see that this homomorphism is well defined, note that for any two holomorphic functions f and g defined around x with $f(x) = 0 = g(x)$ and $\mathrm{d}f(x) = \mathrm{d}g(x)$, the function $f - g$ vanishes of order at least two at x.

The inclusion of $\mathcal{O}_Z(-(k+1)\Delta)$ in $\mathcal{O}_Z(-k\Delta)$ induces an exact sequence of vector bundles

$$0 \longrightarrow K_X^{\otimes k} \otimes E \longrightarrow J^k(E) \longrightarrow J^{k-1}(E) \longrightarrow 0 \tag{2.1}$$

over X. The above homomorphism $K_X^{\otimes k} \otimes E \longrightarrow J^k(E)$ is constructed using the homomorphism $f_{\mathcal{O},k}$ defined above. More precisely, for any $(\mathrm{d}f)^{\otimes k}(x) \in (K_X^{\otimes k})_x$, where f, as above, is a holomorphic function defined around x with $f(x) = 0$, and for any $e \in E_x$ in the fiber of E over x, the image of $(\mathrm{d}f)^{\otimes k}(x) \otimes e$ by the inclusion map in (2.1) is the element in $J^k(E)_x$ representing the locally defined section $f \cdot \widehat{e}$ of E, where $\widehat{e}$ is a holomorphic section of E defined around x with $\widehat{e}(x) = e$. It is easy to check that this element of $J^k(E)_x$ does not depend on the choice of the local section $\widehat{e}$ extending e.

2.2. Differential operators

Let E and F be two holomorphic vector bundles over X. The sheaf of *differential operators* of order k from E to F, which is denoted by $\mathrm{Diff}^k_X(E, F)$, is defined as

$$\mathrm{Diff}^k_X(E, F) := \mathrm{Hom}(J^k(E), F) = J^k(E)^* \otimes F. \tag{2.2}$$

Consider the composition

$$\sigma : \mathrm{Diff}^k_X(E, F) = J^k(E)^* \otimes F \longrightarrow (K_X^{\otimes k} \otimes E)^* \otimes F \tag{2.3}$$

where the right-hand side homomorphism is Id_F tensored with the dual of the injective homomorphism in (2.1). This homomorphism σ is known as the *symbol map*. So we have an exact sequence of vector bundles

$$0 \longrightarrow \mathrm{Diff}^{k-1}_X(E, F) \longrightarrow \mathrm{Diff}^k_X(E, F) \xrightarrow{\sigma} (K_X^{\otimes k} \otimes E)^* \otimes F \longrightarrow 0 \tag{2.4}$$

which is obtained from (2.1).

We will now give an alternative description of the differential operators.

For any $n \geq 0$, consider the quotient (coherent) sheaf

$$\mathcal{F}(n) := \frac{p_2^* K_X \otimes \mathcal{O}_Z((n+1)\Delta)}{p_2^* K_X}$$

over Z. So $\mathcal{F}(n)$ is supported over the nonreduced divisor $(n+1)\Delta$.

Let $U \subset X$ be an analytic open subset and $z : U \longrightarrow \mathbb{C}$ a holomorphic coordinate function on it. We have a homomorphism of sheaves

$$\delta_U(n) : \mathcal{F}(n)|_{p_1^{-1}(U)} \longrightarrow \mathcal{O}_U$$

which is defined as follows: for any holomorphic section

$$s = \frac{f(z_1, z_2)}{(z_2 - z_1)^{n+1}} \mathrm{d}z_2 \in \Gamma(U \times U; \mathcal{F}(n))$$

over $U \times U$, where (z_1, z_2) is the coordinate function on $U \times U$ defined by $(z_1, z_2)(u_1, u_2) = (z(u_1), z(u_2)) \in \mathbb{C}^2$, set

$$\delta_U(n)(s)(x) := \frac{1}{n!} \frac{\partial^n f}{\partial z_2^n}(x, x)$$

for any $x \in U$. It is straight-forward to check that this homomorphism $\delta_U(n)$ does not depend on the choice of the coordinate function z on U. Consequently, these locally defined homomorphisms $\delta_U(n)$ patch together compatibly to define a global homomorphism

$$\delta(n) : p_{1*}\mathcal{F}(n) \longrightarrow \mathcal{O}_X \tag{2.5}$$

of vector bundles over X.

Let E and F be two holomorphic vector bundles over X. Define the coherent sheaf

$$\mathcal{F}(E, F; n) := \frac{p_1^* F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z((n+1)\Delta)}{p_1^* F \otimes p_2^*(K_X \otimes E^*)}$$

over Z. So $\mathcal{F}(E, F; n)$ is again supported over the nonreduced divisor $(n+1)\Delta$, and it is identified with the direct image $\iota_*\iota^*(p_1^*F \bigotimes p_2^*(K_X \bigotimes E^*) \bigotimes \mathcal{O}_Z((n+1)\Delta))$, where ι is the inclusion of $(n+1)\Delta$ in Z.

There is a natural isomorphism

$$\mathcal{K} : H^0((n+1)\Delta, \mathcal{F}(E, F; n)) \longrightarrow H^0(X, \mathrm{Diff}_X^n(E, F)). \tag{2.6}$$

To construct the isomorphism in (2.6), take any $\kappa \in H^0((n+1)\Delta, \mathcal{F}(E, F; n))$, and let u be a holomorphic section of E defined over an open subset U of X. So the contraction $\langle \kappa, p_2^*u \rangle$ is a section of $p_1^*F \bigotimes \mathcal{F}(n)$ over $p_1^{-1}(U)$; the contraction used here is the natural pairing of E with E^*. Therefore, using the projection formula $p_{1*}p_1^*F = F \bigotimes (p_{1*}\mathcal{O}_Z) = F$ we have

$$\delta(n)(\langle \kappa, p_2^*u \rangle) \in \Gamma(U; F),$$

where the homomorphism $\delta(n)$ is defined in (2.5). Finally, define the homomorphism $\mathcal{K}$ in (2.6) to be

$$\mathcal{K}(\kappa)(u) = \delta(n)(\langle \kappa, p_2^*u \rangle).$$

The homomorphism $\mathcal{K}$ constructed this way is clearly an isomorphism.

The Poincaré adjunction formula says that the restriction of the line bundle $\mathcal{O}_Z(\Delta)$ to the divisor Δ is identified with

$$N_\Delta \cong T\Delta \cong (p_i^*TX)|_\Delta$$

where N_Δ is the normal bundle to Δ and $T\Delta$ is the (holomorphic) tangent bundle of Δ. Note that the isomorphism of $T\Delta$ with N_Δ depends on the ordering of $X \times X$; the flip isomorphism $(x, y) \longmapsto (y, x)$ corresponds to multiplying the isomorphism $N_\Delta \cong T\Delta$ with -1 (see [3, page 1315]). Now, the inclusion $\Delta \hookrightarrow (n+1)\Delta$ defines a projection

$$\mathcal{F}(E, F; n) \longrightarrow \mathrm{Hom}(E, F) \otimes (TX)^{\otimes n}$$

where the vector bundle $\mathrm{Hom}(E, F) \otimes (TX)^{\otimes n}$ over X is considered as a sheaf supported over the reduced diagonal $\Delta \subset Z$ using the natural identification of X with Δ defined by $x \longmapsto (x, x)$. Combining this projection with $\mathcal{K}^{-1}$ in (2.6) we get a homomorphism

$$H^0(X, \mathrm{Diff}_X^n(E, F)) \longrightarrow H^0(X, \mathrm{Hom}(E, F) \otimes (TX)^{\otimes n}). \tag{2.7}$$

This homomorphism coincides with the symbol homomorphism σ defined in (2.3).

Consider the de Rham differential (exterior derivative)

$$\mathrm{d} : \mathcal{O}_X \longrightarrow K_X \tag{2.8}$$

which is a differential operator of order one. Using the isomorphism in (2.6), the differential operator d gives a section

$$\varphi_{\mathrm{dr}} \in \Gamma(2\Delta; p_1^*K_X \otimes p_2^*K_X \otimes \mathcal{O}_Z(2\Delta)) \tag{2.9}$$

over the nonreduced diagonal 2Δ. Using the Poincaré adjunction formula, the line bundle $(p_1^*K_X \bigotimes p_2^*K_X \bigotimes \mathcal{O}_Z(2\Delta))|_\Delta$ is canonically trivialized. Since the symbol of the differential operator d is the constant function 1, the restriction of φ_{dr} to Δ coincides with the constant function 1 (in terms of the canonical trivialization of $p_1^*K_X \otimes p_2^*K_X \otimes \mathcal{O}_Z(2\Delta)$ over Δ).

3. Logarithmic connection and orbifold projective structure

3.1. Logarithmic connection

Let

$$\mathbb{D} := \sum_{i=1}^{\ell} \zeta_i$$

be a reduced divisor on the compact Riemann surface X. So ζ_i are distinct ℓ points on X. We *do not* assume that $\ell \neq 0$.

Let E be a holomorphic vector bundle over X. A *logarithmic connection* on E singular over $\mathbb{D}$ is a first order differential operator

$$\nabla : E \longrightarrow K_X \otimes \mathcal{O}_X(\mathbb{D}) \otimes E$$

satisfying the Leibniz identity which says that

$$\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s$$

where s (respectively, f) is any locally defined holomorphic section of E (respectively, holomorphic function over X). Note that any logarithmic connection on a Riemann surface is automatically flat as there are no nonzero holomorphic 2-forms on it.

The above condition that ∇ satisfies the Leibniz identity is clearly equivalent to the condition that the symbol of the differential operator ∇ coincides with

$$\mathrm{Id}_E \in H^0(X, \mathcal{O}_X(\mathbb{D}) \otimes \mathrm{End}(E)),$$

where Id_E denotes the identity automorphism of E.

Let $v \in E_{\zeta_i}$ be a vector in the fiber of E over $\zeta_i \in \mathbb{D}$. Let $\widehat{v}$ be any holomorphic section of E defined around ζ_i such that $\widehat{v}(\zeta_i) = v$. Consider

$$\nabla(\widehat{v})(\zeta_i) \in (K_X \otimes \mathcal{O}_X(\mathbb{D}))_{\zeta_i} \otimes_{\mathbb{C}} E_{\zeta_i} = \mathbb{C} \otimes_{\mathbb{C}} E_{\zeta_i} = E_{\zeta_i}$$

with $(K_X \bigotimes \mathcal{O}_X(\mathbb{D}))_{\zeta_i}$ being identified with $\mathbb{C}$ using the Poincaré adjunction formula. Note that if $v = 0$, then $\nabla(\widehat{v})$ is a (locally defined) section of $K_X \bigotimes E$. So, in that case the evaluation $\nabla(\widehat{v})(\zeta_i) \in E_{\zeta_i}$ vanishes. Using this it follows that $\nabla(\widehat{v})(\zeta_i)$ is independent of the choice of the section $\widehat{v}$ extending v. Consequently, we have a well-defined endomorphism

$$\mathrm{Res}(\nabla, \zeta_i) \in \mathrm{End}(E_{\zeta_i})$$

that sends any $v \in E_{\zeta_i}$ to $\nabla(\widehat{v})(\zeta_i)$. This endomorphism $\mathrm{Res}(\nabla, \zeta_i)$ is called the *residue* of the logarithmic connection ∇ at the point ζ_i.

Take a point

$$x_0 \in X' := X \setminus \mathbb{D}$$

in the complement, and let $\gamma_i \in \pi_1(X', x_0)$ be an element defined by a positively oriented loop around ζ_i. In other words, take a smooth orientation preserving diffeomorphism f of the closed unit disk in $\mathbb{C}$ to $X' \cup \{\zeta_i\}$ such that $f(1) = x_0$ and $f(0) = \zeta_i$. The image, under the map f, of the unit circle in $\mathbb{C}$ with its anti-clockwise orientation represents γ_i. Let

$$A_i \in \mathrm{End}(E_{x_0})$$

be the monodromy of the flat connection ∇ for γ_i. This automorphism A_i is conjugate to $\exp(-2\pi\sqrt{-1}\text{Res}(\nabla, \zeta_i))$, that is, there is an isomorphism of E_{x_0} with E_{ζ_i} that takes A_i to $\exp(-2\pi\sqrt{-1}\text{Res}(\nabla, \zeta_i))$ [8, page 79, Proposition 3.11].

Consider the vector bundle $\text{Diff}^1_X(E, E)$ over X. The exact sequence (2.4) becomes

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{Diff}^1_X(E, E) \stackrel{\sigma}{\longrightarrow} TX \otimes \text{End}(E) \longrightarrow 0 \tag{3.1}$$

where TX is the holomorphic tangent bundle of X. The vector bundle $TX \bigotimes_{\mathcal{O}_X} \text{End}(E)$ has a holomorphic line subbundle defined by $TX \bigotimes_{\mathbb{C}} \text{Id}_E$, which is identified with TX. Let

$$f_0 : TX \otimes \mathcal{O}_X(-\mathbb{D}) \longrightarrow TX \tag{3.2}$$

be the natural inclusion homomorphism. The vector bundle

$$\text{At}(E) := \sigma^{-1}(f_0(TX \otimes \mathcal{O}_X(-\mathbb{D})) \otimes \text{Id}_E) \subset \text{Diff}^1_X(E, E)$$

is called the *Atiyah bundle*, where f_0 and σ are defined in (3.2) and (3.1) respectively [1]. So (3.1) gives an exact sequence

$$0 \longrightarrow \text{End}(E) \longrightarrow \text{At}(E) \longrightarrow TX \otimes \mathcal{O}_X(-\mathbb{D}) \longrightarrow 0 \tag{3.3}$$

which is known as the *Atiyah exact sequence*.

Giving a logarithmic connection on a holomorphic vector bundle E is equivalent to giving a holomorphic splitting of the Atiyah exact sequence constructed in (3.3). Indeed, a splitting of the Atiyah exact sequence gives a homomorphism from $TX \bigotimes \mathcal{O}_X(-\mathbb{D})$ to $\text{Diff}^1_X(E, E)$. This homomorphism using the natural isomorphism

$$\text{Diff}^1_X(E, E) \otimes (TX \otimes \mathcal{O}_X(-\mathbb{D}))^* \cong \text{Diff}^1_X(E, K_X \otimes \mathcal{O}_X(\mathbb{D}) \otimes E)$$

gives a differential operator defining a logarithmic connection on E. Therefore, a logarithmic connection on E is a holomorphic splitting of the Atiyah exact sequence.

Consider the holomorphic section over 2Δ given by (2.6) for a differential operator defining a logarithmic connection on E. Contracting this section with the dual of the section in (2.9) we get a section of $p_1^*(\mathcal{O}_X(\mathbb{D}) \bigotimes E) \bigotimes p_2^* E^*$ over 2Δ. Using this construction, giving a logarithmic connection on E is equivalent to giving a section of $p_1^*(\mathcal{O}_X(\mathbb{D}) \bigotimes E) \bigotimes p_2^* E^*$ over 2Δ whose restriction to Δ coincides with the section defined by Id_E. This description of a logarithmic connection is due to A. Grothendieck.

Now we will recall the definition of a second fundamental form.

Let E be a holomorphic vector bundle over X equipped with a logarithmic connection ∇ and F a holomorphic subbundle of E. Consider the composition

$$F \hookrightarrow E \stackrel{\nabla}{\longrightarrow} K_X \otimes \mathcal{O}_X(\mathbb{D}) \otimes E \stackrel{\text{Id}\otimes q}{\longrightarrow} K_X \otimes \mathcal{O}_X(\mathbb{D}) \otimes (E/F)$$

where q is the natural projection of E to E/F. The Leibniz identity ensures that the above composition homomorphism is $\mathcal{O}_X$-linear. In other words, we have a homomorphism of vector bundles

$$S(\nabla, F) \in H^0(X, \text{Hom}(F, K_X \otimes \mathcal{O}_X(\mathbb{D}) \otimes (E/F))) \tag{3.4}$$

over X. This homomorphism $S(\nabla, F)$ is called the *second fundamental form* of the subbundle F for the connection ∇.

3.2. Orbifold projective structure

Fix a function

$$\varpi : \mathbb{D} \longrightarrow \mathbb{N}^+ \setminus \{1\}, \tag{3.5}$$

from the given subset $\mathbb{D} \subset X$.

Let $\{U_i\}_{i\in I}$ be a covering of X by connected open subsets. Assume that $\#\mathbb{D} \cap U_i \leq 1$ for each $i \in I$, that is, each U_i contains at most one point from $\mathbb{D}$. If $\mathbb{D} \cap U_i = \emptyset$, by a *holomorphic coordinate function* on U_i we will mean an injective holomorphic map ϕ_i from an open subset of $\mathbb{CP}^1$ to U_i, that is, a holomorphic isomorphism

$$\phi_i : V_i \longrightarrow U_i,$$

where V_i is a connected open subset of $\mathbb{CP}^1$. If $\mathbb{D} \cap U_i = \zeta_j$, then by a *holomorphic coordinate function* on U_i we will mean a holomorphic Galois (ramified) covering map

$$\phi_i : V_i \longrightarrow U_i \tag{3.6}$$

from some connected open subset $V_i \subset \mathbb{CP}^1$ such that

(1) the degree of ϕ_i is $\varpi(\zeta_j)$, where ϖ is the function in (3.5);
(2) the Galois group of the covering ϕ_i is the cyclic group $\mathbb{Z}/\varpi(\zeta_j)\mathbb{Z}$;
(3) the map ϕ_i is unramified over $U_i \setminus \{\zeta_j\}$, and it is totally ramified over ζ_j (the inverse image of ζ_j is a single point).

Recall that the Möbius group $\mathrm{PSL}(2, \mathbb{C})$ is the group of all holomorphic automorphisms of $\mathbb{CP}^1$. An orbifold projective structure on X is defined by giving a covering $\{U_i, \phi_i\}_{i\in I}$ by holomorphic coordinate functions (defined above) such that

(1) if $\mathbb{D} \cap U_i = \zeta_j$, then each deck transformation of the Galois covering map ϕ_i in (3.6) coincides with the restriction of a Möbius transformation to V_i;
(2) for each pair $i, i' \in I$ and every connected simply connected open subset $V \subset \phi_{i'}^{-1}((U_i \cap U_{i'}) \setminus \mathbb{D})$, each branch of $\phi_i^{-1} \circ \phi_{i'}$ over V coincides with the restriction of some Möbius transformation.

By a branch of $\phi_i^{-1} \circ \phi_{i'}$ we mean a holomorphic function $f : V \longrightarrow \mathbb{CP}^1$ such that $\phi_{i'} = \phi_i \circ f$ on V.

Note that the second condition actually implies the first condition by setting $i = i'$. The first condition implies that if one branch of $\phi_i^{-1} \circ \phi_{i'}$ over V coincides with the restriction of some Möbius transformation, then every branch of $\phi_i^{-1} \circ \phi_{i'}$ over V coincides with the restriction of some Möbius transformation.

Definition 3.1. Two such data $\{U_i, \phi_i\}_{i\in I}$ and $\{U_i, \phi_i\}_{i\in I'}$ satisfying all the above conditions are called *equivalent* if their union $\{U_i, \phi_i\}_{i\in I\cup I'}$ also satisfies all the above conditions. An *orbifold projective structure* on X is an equivalence class of such data.

If $\mathbb{D} = \emptyset$, then an orbifold projective structure on X is called a *projective structure* (see [9,8]).

The following lemma says when X admits an orbifold projective structure.

Lemma 3.2. *If $\ell := \#D = 0$ (that is, $\mathbb{D} = \emptyset$), then X admits an orbifold projective structure.*

If $\ell \geq 1$, then X admits an orbifold projective structure if and only if at least one of the following three conditions holds:

(1) $\text{genus}(X) \geq 1$;
(2) $\ell \geq 3$ *and* $\text{genus}(X) = 0$;
(3) $\ell = 2$, $\text{genus}(X) = 0$ *and* $\varpi(\zeta_1) = \varpi(\zeta_2)$.

In other words, X does not admit an orbifold projective structure if and only if either $\text{genus}(X) = 0 = \ell - 1$, *or* $\text{genus}(X) = 0 = \ell - 2$ *with* $\varpi(\zeta_1) \neq \varpi(\zeta_2)$.

Proof. The uniformization theorem says that the universal cover $\widetilde{X}$ of X is biholomorphic to either $\mathbb{C}$ or $\mathbb{CP}^1$ or the upper half plane $\mathbb{H}$. Consequently, the group of all holomorphic automorphisms $\text{Aut}(\widetilde{X})$ is contained in $\text{PSL}(2, \mathbb{C}) = \text{Aut}(\mathbb{CP}^1)$. Therefore the uniformization theorem gives a natural projective structure on X if $\ell = 0$.

Assume that $\ell \geq 1$, and also assume that one of the three conditions in the statement of the lemma holds. Under this assumption, a theorem due to Bundgaard-Nielsen and Fox says that there is a finite Galois covering

$$\gamma : Y \longrightarrow X \tag{3.7}$$

such that γ is ramified exactly over the divisor $\mathbb{D}$ and, furthermore, the order of ramification over each point $\zeta_i \in \mathbb{D}$ is $\varpi(\zeta_i)$ [12, page 26, Proposition 1.2.12]. A clarification about Proposition 1.2.12 of [12] is needed. The way Proposition 1.2.12 of [12] is stated it seems to mean that the order of ramification over each ζ_i is a multiple of $\varpi(\zeta_i)$. However, the proof of the proposition shows that the order of ramification over each ζ_i is exactly $\varpi(\zeta_i)$. See the last three lines in page 27 of [12]; from there it follows that the order of ramification over any ζ_i is $\varpi(\zeta_i)$.

Fix a projective structure P on the compact Riemann surface Y in (3.7). For any holomorphic automorphism F of Y, the pullback F^*P is a projective structure on Y. We know that the space of all projective structures on Y is an affine space for the vector space of all quadratic differentials $H^0(Y, K_Y^{\otimes 2})$ over Y [9, page 170, Theorem 19], [8, page 32, Proposition 5.8].

Let

$$G := \text{Gal}(\gamma)$$

be the Galois group for the covering γ in (3.7). Consider the convex combination

$$\mathcal{P} := \frac{\sum_{F \in G} F^*P}{\#G}$$

($\#G$ is the order of G), where the average is defined using the convex structure of the space of all projective structures on Y. This projective structure $\mathcal{P}$ on Y is clearly left invariant by the action of G on Y. We will construct an orbifold projective structure on X using $\mathcal{P}$.

Let U be a connected simply connected open subset of Y left invariant by the action of G on Y and

$$\phi : V \longrightarrow U$$

a holomorphic isomorphism with $V \subset \mathbb{CP}^1$ compatible with the projective structure $\mathcal{P}$ on Y. Consider the composition

$$\gamma \circ \phi : V \longrightarrow \gamma(U) \subset X.$$

All functions of the form $\gamma \circ \phi$ obtained this way combine together to define an orbifold projective structure on X. Indeed, that they define an orbifold projective structure on X is an immediate consequence of the facts that $\mathcal{P}$ is left invariant by the action on Y of the Galois group G and γ is ramified exactly over $\mathbb{D}$ with $\varpi(\zeta_i)$ as the order of ramification over each $\zeta_i \in \mathbb{D}$.

If $\text{genus}(X) = 0$ and $\ell = 1$, then the complement $X' := X \setminus \mathbb{D}$ is simply connected. So, in that case if P is an orbifold projective structure on X, then there is a holomorphic coordinate function compatible with P that defines an isomorphism of X' with $\mathbb{C} := \mathbb{CP}^1 \setminus \{\infty\}$. Therefore, this holomorphic coordinate function extends to a holomorphic map from $\mathbb{CP}^1$ to X which is ramified exactly over $\zeta_1 = \mathbb{D}$ with the order of ramification being $\varpi(\zeta_1)$. Since such a map does not exist (recall that $\varpi(\zeta_1) > 1$), we conclude that X does not admit an orbifold projective structure.

If $\text{genus}(X) = 0$ and $\ell = 2$, then $\pi_1(X') = \mathbb{Z}$. Hence the simple loops around ζ_1 and ζ_2 are homotopic (with opposite orientation). From this it follows that if X admits an orbifold projective structure, then $\varpi(\zeta_1) = \varpi(\zeta_2)$. This completes the proof of the lemma. □

Henceforth we will always assume that one of the following is valid:

(1) $\ell = 0$;
(2) $\ell \geq 1$ and $\text{genus}(X) \geq 1$;
(3) $\ell \geq 3$ and $\text{genus}(X) = 0$;
(4) $\text{genus}(X) = 0$ with $\ell = 2$ and $\varpi(\zeta_1) = \varpi(\zeta_2)$.

So, by Lemma 3.2, the Riemann surface X admits an orbifold projective structure.

Lemma 3.3. *The space of all orbifold projective structures on the Riemann surface X is an affine space for the vector space $H^0(X, \mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2})$, the space of all meromorphic quadratic differentials on X with at most simple poles at the points of the divisor $\mathbb{D}$.*

Proof. As was noted earlier, the space of all orbifold projective structures on the Riemann surface X is nonempty. If $\ell = 0$, then the lemma is well known [8, page 32, Proposition 5.8]. So assume that $\ell \geq 1$.

Let P_1 and P_2 be two orbifold projective structures on X. So, over the complement $X' := X \setminus \mathbb{D}$, the restrictions of P_1 and P_2 (to X') differ by a holomorphic section

$$\theta \in H^0(X', K_X^{\otimes 2})$$

[8, page 32, Proposition 5.8]. We will show that θ extends to a holomorphic section of $\mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2}$ over X.

Fix a covering γ as in (3.7). We will show that the orbifold projective structure P_i, $i = 1, 2$, on X gives a projective structure $\overline{P}_i$ on the covering surface Y. The projective structure $\overline{P}_i$ is in fact defined as done in the proof of Lemma 3.2. In other words, if $\phi : V \longrightarrow U$ is a holomorphic coordinate function from a connected simply connected open subset $V \subset \mathbb{CP}^1$ to $U \subset X$ compatible with respect to the orbifold projective structure P_i, then ϕ lifts to a biholomorphic map

$$\overline{\phi} : V \longrightarrow \gamma^{-1}(U)^0 \subset \gamma^{-1}(U)$$

such that $\gamma \circ \overline{\phi} = \phi$, where $\gamma^{-1}(U)^0$ is any connected component of $\gamma^{-1}(U)$. The existence of such $\overline{\phi}$ follows from the fact that the ramifications of $\gamma|_{\gamma^{-1}(U)^0}$ and ϕ are identical. The holomorphic coordinate functions $\overline{\phi}$ obtained this way define the projective structure $\overline{P}_i$ on Y. From this construction of $\overline{P}_i$ it is immediate that $\overline{P}_i$ is left invariant by the action of the Galois group G on Y.

So $\overline{P}_1$ and $\overline{P}_2$ differ by

$$\overline{\theta} \in H^0(Y, K_Y^{\otimes 2})^G \subset H^0(Y, K_Y^{\otimes 2}),$$

where $H^0(Y, K_Y^{\otimes 2})^G$ denotes the space of all quadratic differentials on Y that are invariant under the action of G on Y. Note that over $\gamma^{-1}(X')$ we have $\overline{\theta} = \gamma^*\theta$.

Let $D := \{z \in \mathbb{C} \mid |z|^2 < 1\}$ be the open unit disk and $\psi(z) = z^k$ the degree k self-map of D, where $k \geq 1$. If ω is a quadratic differential on D invariant under the action of the Galois group $\mathbb{Z}/k\mathbb{Z}$ for ψ, then ω descends, by the map ψ, to a quadratic differential with at most a simple pole at 0. In other words, $\omega = \psi^*\omega'$, where ω' is a meromorphic quadratic differential on D with pole only at 0 of order at most one. Indeed, this follows immediately from the fact that an invariant quadratic differential ω on the disk must be of the form $z \longmapsto f(z^k)z^{k-2}\mathrm{d}z^{\otimes 2}$, where f is a holomorphic function on D.

From the above observation it follows immediately that

$$H^0(Y, K_Y^{\otimes 2})^G = H^0(X, \mathcal{O}_X(\mathbb{D}) \otimes K_X^{\otimes 2}). \tag{3.8}$$

In particular, the holomorphic quadratic differential θ on $X' = X \setminus \mathbb{D}$ extends to X as a holomorphic section of $\mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2}$. This extended section over X corresponds to $\overline{\theta}$ by the isomorphism in (3.8). Therefore, any two orbifold projective structures on X differ by a holomorphic section of $\mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2}$.

For the converse direction, we first recall that the space of all orbifold projective structures on X is nonempty. Now, for any $\omega \in H^0(X, \mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2})$, the isomorphism in (3.8) gives $\overline{\omega} \in H^0(Y, K_Y^{\otimes 2})^G$, a G-invariant quadratic differential on Y. So using the affine space structure of the space of all projective structures on Y, the projective structure $\overline{P}_1$ on Y constructed earlier from P_1 and the quadratic differential $\overline{\omega}$ on Y together give a projective structure $\overline{P}$ on Y. Since both $\overline{P}_1$ and $\overline{\omega}$ are G-invariant, the projective structure $\overline{P}$ is also left invariant by the action of G on Y. Therefore, $\overline{P}$ gives an orbifold projective structure P on X whose construction is described in the proof of Lemma 3.2.

Sending any pair (P_1, ω) to P we conclude that the space of all orbifold projective structures on X is an affine space for the vector space $H^0(X, \mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2})$. This completes the proof of the lemma. □

The above lemma implies that the space of all orbifold projective structures on X is a complex affine space of dimension

(1) $3(\text{genus}(X) - 1) + \#\mathbb{D}$ if $\text{genus}(X) > 1$;
(2) $\#\mathbb{D}$ if $\text{genus}(X) = 1$ and $\#\mathbb{D} \geq 1$;
(3) 1 if $\text{genus}(X) = 1$ and $\#\mathbb{D} = 0$;
(4) $\#\mathbb{D} - 3$ if $\text{genus}(X) = 0$ and $\#\mathbb{D} \geq 4$;
(5) 0 if $\text{genus}(X) = 0$ and $\#\mathbb{D} \leq 3$.

4. Projective structure and connection

In this section we will describe projective structures on a compact Riemann surface using connections. Throughout this section $\mathbb{D}$ will be the empty set (the zero divisor).

A holomorphic connection ∇ on a rank two holomorphic vector bundle V over X is called a $\mathrm{SL}(2, \mathbb{C})$-connection if the monodromy of ∇ is contained in $\mathrm{SL}(2, \mathbb{C})$. So the line bundle $\bigwedge^2 V$ is trivial if V admits a $\mathrm{SL}(2, \mathbb{C})$-connection.

A $\mathrm{SL}(2, \mathbb{C})$-*structure* on X is a triple (V, ∇, ξ), where V of rank two holomorphic vector bundle over X equipped with a $\mathrm{SL}(2, \mathbb{C})$-connection ∇ and $\xi \subset V$ a holomorphic line subbundle such that the second fundamental form (defined in (3.4))

$$\xi \longrightarrow K_X \otimes (V/\xi)$$

is an isomorphism. Since $\bigwedge^2 V \cong \mathcal{O}_X$, this implies that $\xi^{\otimes 2} \cong K_X$, that is, ξ is a theta characteristic on X.

A SL(2, $\mathbb{C}$)-structure gives a projective structure as follows. For any connected simply connected open subset $U \subset X$, using the connection ∇ the restricted vector bundle $V|_U$ can be trivialized. Once we fix such a trivialization, the line subbundle $\xi|_U$ defines a holomorphic map of U to $\mathbb{CP}^1$. The above condition on the second fundamental form ensures that this is an embedding. Using these maps as coordinate charts, a projective structure on X is obtained.

Every projective structure comes from some SL(2, $\mathbb{C}$)-structure. Given any projective structure P on X, the space of all SL(2, $\mathbb{C}$)-structures on X that give rise to P is in a natural bijective correspondence with the space of all theta characteristics on X [9, page 193, Lemma 28]. The theta characteristic corresponding to a SL(2, $\mathbb{C}$)-structure (V, ∇, ξ) is ξ. There are exactly 2^{2g} theta characteristics on X, where g is the genus of X.

For a SL(2, $\mathbb{C}$)-structure (V, ∇, ξ) as above, we have $V \cong J^1(V/\xi)$. To construct the isomorphism, take any point $x \in X$ and $v \in V_x$. Let s_v be the unique flat section (for the connection ∇) of V defined in a neighborhood of x such that $s_v(x) = v$. Let $v' \in J^1(V/\xi)_x$ be the vector defined by the section $q(s_v)$ of V/ξ, where $q : V \longrightarrow V/\xi$ is the quotient map. The isomorphism $V \cong J^1(V/\xi)$ is defined by sending any v to v' constructed above.

Conversely, for any holomorphic line bundle ϑ over X with $\vartheta^{\otimes 2} \cong K_X^{-1} = TX$, the jet bundle $J^1(\vartheta)$ admits holomorphic connections with monodromy contained in SL(2, $\mathbb{C}$) (note that $\bigwedge^2 J^1(\vartheta) \cong \mathcal{O}_X$). Furthermore, any SL(2, $\mathbb{C}$)-connection on $J^1(\vartheta)$ defines a SL(2, $\mathbb{C}$)-structure on X, provided $g \neq 1$.

Although the above description of a projective structure using connection involved the choice of a theta characteristic, the constructions can be suitably modified to get rid of such a choice. This will be explained below.

Let ζ be a holomorphic line bundle over X with $\zeta^{\otimes 2} \cong \mathcal{O}_X$. The line bundle ζ has a natural homomorphic connection ∇^ζ. A (locally defined) section s of ζ is flat with respect to ∇^ζ if and only $s \otimes s$ is a constant function. Note that after fixing an isomorphism of $\zeta^{\otimes 2}$ with the trivial line bundle, $s \otimes s$ gives a holomorphic function; the condition that this is a constant function does not depend on the choice of the isomorphism. This condition on ∇^ζ determines the connection ∇^ζ uniquely.

Let W be a holomorphic vector bundle over X. Using the connection ∇^ζ we have a natural isomorphism

$$J^i(W) \otimes \zeta \cong J^i(W \otimes \zeta) \tag{4.1}$$

for each $i \geq 1$. To construct this isomorphism, note that given any holomorphic section s' of $W \otimes \zeta$ over a connected simply connected open subset of X, we have

$$s' = s_0 \otimes s,$$

where s_0 is a holomorphic section of W and s is a flat section of ζ. Now, $s_0 \otimes s$ defines a section of $J^i(W) \bigotimes \zeta$. Since any two flat sections of ζ over a connected simply connected open subset of X differ by multiplication with a constant scalar, it follows immediately that the homomorphism $J^i(W \bigotimes \zeta) \longrightarrow J^i(W) \bigotimes \zeta$ that sends the section of $J^i(W \bigotimes \zeta)$ defined by s' to the section of $J^i(W) \bigotimes \zeta$ defined by $s_0 \otimes s$ is well defined. This homomorphism evidently is an isomorphism, and this is the isomorphism in (4.1).

Consequently, using (4.1) we have a canonical isomorphism

$$\mathrm{End}(J^i(W \otimes \zeta)) \cong \mathrm{End}(J^i(W) \otimes \zeta) = \mathrm{End}(J^i(W)).$$

This isomorphism induces an isomorphism

$$\mathrm{ad}(J^i(W \otimes \zeta)) \cong \mathrm{ad}(J^i(W) \otimes \zeta) = \mathrm{ad}(J^i(W)),$$

where $\mathrm{ad}(J^i(W \otimes \zeta)) \subset \mathrm{End}(J^i(W \otimes \zeta))$ and $\mathrm{ad}(J^i(W)) \subset \mathrm{End}(J^i(W))$ are the subbundles defined by trace zero endomorphisms.

Consequently, for a $\mathrm{SL}(2, \mathbb{C})$-structure (V, ∇, ξ), the holomorphic vector bundle $\mathrm{ad}(V)$ does not depend on the choice of the $\mathrm{SL}(2, \mathbb{C})$-structure. More precisely, as $V = J^1(V/\xi)$, if (V', ∇', ξ') is another $\mathrm{SL}(2, \mathbb{C})$-structure, then the vector bundle $\mathrm{ad}(V)$ is canonically isomorphic to $\mathrm{ad}(V')$.

Let V_0 be a holomorphic vector bundle over X. Let $\mathrm{ad}(V_0) \subset \mathrm{End}(V_0)$ be the subbundle defined by trace zero endomorphisms. For any integer $k \geq 1$, we denote by $\mathrm{Sym}^k(V_0)$ the vector bundle defined by the k-th symmetric power. If V_0 is of rank two with $\bigwedge^2 V_0 \cong \mathcal{O}_X$, and if we fix a trivialization of $\bigwedge^2 V_0$, then the vector bundle $\mathrm{Sym}^2(V_0)$ is canonically isomorphic to $\mathrm{ad}(V_0)$. Indeed, a trivialization of $\bigwedge^2 V_0$ gives a nowhere vanishing section $s \in H^0(X, \bigwedge^2 V_0^*)$. Now for any $w \in \mathrm{Sym}^2(V_0)_x$, we have $w' \in \mathrm{ad}(V_0)_x$ defined by

$$w'(v) = \langle \langle s(x), v \rangle, w \rangle \in (V_0)_x,$$

for all $v \in (V_0)_x$; here $\langle -, - \rangle$ denotes the contraction using duality pairing. The homomorphism defined by $w \longrightarrow w'$ is an isomorphism of $\mathrm{Sym}^2(V_0)$ with $\mathrm{ad}(V_0)$.

Let (V, ∇, ξ) be a $\mathrm{SL}(2, \mathbb{C})$-structure. So $\bigwedge^2 V$ is isomorphic to the trivial line bundle over X. Fix a trivialization of this line bundle. The above remark shows that $\mathrm{Sym}^2(V) \cong \mathrm{ad}(V)$. The following proposition shows that there are isomorphisms

$$\mathrm{Sym}^2(V) \cong J^2(TX) \cong \mathrm{ad}(V).$$

Proposition 4.1. *A projective structure on X induces an isomorphism of $J^2(TX)$ with $\mathrm{ad}(J^1(\xi^*))$, where ξ is any theta characteristic on X.*

Proof. Using the canonical isomorphism in (4.1) we already know that if ξ and ξ_1 are two theta characteristics on X, then $\mathrm{ad}(J^1(\xi^*))$ is canonically isomorphic to $\mathrm{ad}(J^1(\xi_1^*))$.

Let W_0 be a complex vector space of dimension two. Let $\mathrm{sl}(W_0) \subset \mathrm{End}(W_0)$ be the subspace of trace zero endomorphisms, which is the Lie algebra of $\mathrm{SL}(W_0)$.

Let $\mathbb{P}(W_0)$ be the projective line parametrizing all one-dimensional quotient spaces of W_0. Consider the induced action of $\mathrm{SL}(W_0)$ on $\mathbb{P}(W_0)$. Using this action, an element in the Lie algebra $\mathrm{sl}(W_0)$ gives a holomorphic vector field on $\mathbb{P}(W_0)$. In other words, we have a homomorphism

$$f : \mathrm{sl}(W_0) \longrightarrow H^0(\mathbb{P}(W_0), T\mathbb{P}(W_0)) \tag{4.2}$$

which is in fact an isomorphism.

Using f in (4.2), the jet bundle $J^2(T\mathbb{P}(W_0))$ gets identified with the trivial vector bundle $\mathbb{P}(W_0) \times \mathrm{sl}(W_0)$ over $\mathbb{P}(W_0)$ with fiber $\mathrm{sl}(W_0)$. To explain this, first note that there is a natural homomorphism f_0 from the trivial vector bundle over $\mathbb{P}(W_0)$ with fiber $H^0(\mathbb{P}(W_0), T\mathbb{P}(W_0))$ to the vector bundle $J^2(T\mathbb{P}(W_0))$. This homomorphism f_0 is defined by restricting global sections of $T\mathbb{P}(W_0)$ to the second order infinitesimal neighborhood of points of $\mathbb{P}(W_0)$. Since $\mathrm{degree}(T\mathbb{P}(W_0)) = 2$, a section of $T\mathbb{P}(W_0)$ vanishing on the second order infinitesimal neighborhood of any given point actually vanishes identically. From this it follows immediately that the above homomorphism f_0 is an isomorphism. Now using $f_0 \circ f$ the jet bundle $J^2(T\mathbb{P}(W_0))$ gets identified with the trivial vector bundle over $\mathbb{P}(W_0)$ with fiber $\mathrm{sl}(W_0)$, where f is defined in (4.2).

Let

$$I_{\mathbb{P}} : J^2(T\mathbb{P}(W_0)) \longrightarrow \mathbb{P}(W_0) \times \mathrm{sl}(W_0) \tag{4.3}$$

be the isomorphism of vector bundles over $\mathbb{P}(W_0)$ constructed above.

Consider the natural action of the automorphism group

$$\mathrm{Aut}(\mathbb{P}(W_0)) = \mathrm{PGL}(W_0) = \mathrm{GL}(W_0)/\mathbb{C}^*$$

on $\mathbb{P}(W_0)$. The action lifts to an action on $J^2(T\mathbb{P}(W_0))$ in an obvious way. Equip the vector bundle $\mathbb{P}(W_0) \times \mathrm{sl}(W_0)$ with the diagonal action of $\mathrm{Aut}(\mathbb{P}(W_0))$ with $\mathrm{Aut}(\mathbb{P}(W_0)) = \mathrm{PGL}(W_0)$ acting on its Lie algebra $\mathrm{sl}(W_0)$ through inner conjugations. The isomorphism $I_{\mathbb{P}}$ in (4.3) evidently commutes with the actions of $\mathrm{Aut}(\mathbb{P}(W_0))$ on the two vector bundles.

Let $\mathcal{O}_{\mathbb{P}(W_0)}(1)$ be the tautological line bundle over $\mathbb{P}(W_0)$ whose fiber over a point of $\mathbb{P}(W_0)$ is the quotient line represented by the point. The jet bundle $J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1))$ is identified with the trivial vector bundle $\mathbb{P}(W_0) \times W_0$ over $\mathbb{P}(W_0)$ with fiber W_0. Indeed, we have

$$H^0(\mathbb{P}(W_0), \mathcal{O}_{\mathbb{P}(W_0)}(1)) = W_0,$$

and the isomorphism of $\mathbb{P}(W_0)\times W_0$ with $J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1))$ is obtained by restricting global sections of $\mathcal{O}_{\mathbb{P}(W_0)}(1)$ to the first order infinitesimal neighborhood of any given point of $\mathbb{P}(W_0)$.

The above isomorphism of $J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1))$ with $\mathbb{P}(W_0) \times W_0$ gives an isomorphism of $\mathrm{ad}(J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1)))$ with $\mathbb{P}(W_0) \times \mathrm{sl}(W_0)$. Combining this isomorphism with the isomorphism $I_{\mathbb{P}}$ in (4.3) we obtain an isomorphism

$$I'_{\mathbb{P}} : J^2(T\mathbb{P}(W_0)) \longrightarrow \mathrm{ad}(J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1))). \tag{4.4}$$

Note that $\mathrm{GL}(W_0)$ has a natural action on $\mathcal{O}_{\mathbb{P}(W_0)}(1)$. The induced action of $\mathrm{GL}(W_0)$ on $\mathrm{ad}(J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1)))$ clearly descends to an action of $\mathrm{PGL}(W_0)$ on $\mathrm{ad}(J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1)))$. The isomorphism in (4.4) evidently commutes with the actions of $\mathrm{PGL}(W_0)$ on the two vector bundles.

Fix a projective structure on X, and fix a theta characteristic ξ on X. Also, fix an isomorphism $\mathcal{O}_{\mathbb{P}(W_0)}(2) \cong T\mathbb{P}(W_0)$. Let

$$\phi : \mathbb{P}(W_0) \supset V_1 \longrightarrow U_1 \subset X$$

be a biholomorphism as in (3.6) compatible with the given projective structure. Fix an isomorphism

$$\gamma : \xi^*|_{U_1} \longrightarrow \phi^*\mathcal{O}_{\mathbb{P}(W_0)}(1)$$

such that

$$\gamma \otimes \gamma = \mathrm{d}\gamma : TU_1 \longrightarrow \phi^*T\mathbb{P}(W_0);$$

recall that $\xi^* \otimes \xi^* = TX$ and $\mathcal{O}_{\mathbb{P}(W_0)}(2) \cong T\mathbb{P}(W_0)$. Note that there are exactly two choices, namely $\pm\gamma$, that satisfy this condition on γ.

The above isomorphism γ induces an isomorphism

$$J^1(\gamma)' : \mathrm{ad}(J^1(\xi^*))|_{U_1} \longrightarrow \phi^*\mathrm{ad}(J^1(\mathcal{O}_{\mathbb{P}(W_0)}(1))). \tag{4.5}$$

Since the differential $\mathrm{d}\gamma : TU_1 \longrightarrow \phi^*T\mathbb{P}(W_0)$ is an isomorphism, it induces an isomorphism

$$J^2(\mathrm{d}\gamma) : J^2(TX)|_{U_1} \longrightarrow \phi^*J^2(T\mathbb{P}(W_0)).$$

Therefore, we have an isomorphism

$$\widehat{\gamma} := (J^1(\gamma)')^{-1}\phi^* I'_{\mathbb{P}} \circ J^2(\mathrm{d}\gamma) : J^2(TX)|_{U_1} \longrightarrow \mathrm{ad}(J^1(\xi^*))|_{U_1}, \tag{4.6}$$

where $I'_{\mathbb{P}}$ and $J^1(\gamma)'$ are constructed in (4.4) and (4.5) respectively.

Since the isomorphism $I'_{\mathbb{P}}$ in (4.4) is equivariant for the actions of $\mathrm{PGL}(W_0)$, and any two choices of γ differ by multiplication with ± 1, it follows immediately that the isomorphism $\widehat{\gamma}$ does not depend neither on the choice of the coordinate function ϕ (compatible with the given projective structure) nor on the choice of γ. Also, $\widehat{\gamma}$ does not depend on the choice of the isomorphism $\mathcal{O}_{\mathbb{P}(W_0)}(2) \cong T\mathbb{P}(W_0)$.

Consequently, the locally constructed isomorphisms $\widehat{\gamma}$ in (4.6) patch together compatibly to define a global isomorphism

$$I_X : J^2(TX) \longrightarrow \mathrm{ad}(J^1(\xi^*))$$

over X. This completes the proof of the proposition. □

Let ξ be a theta characteristic on X. We noted earlier in this section that a projective structure on X gives a holomorphic connection ∇ on $J^1(\xi^*)$ so that $(J^1(\xi^*), \nabla, \xi)$ defines a $\mathrm{SL}(2,\mathbb{C})$-structure. The connection ∇ on $J^1(\xi^*)$ induces a holomorphic connection on $\mathrm{ad}(J^1(\xi^*))$.

Proposition 4.2. *A projective structure on X gives a holomorphic connection on the jet bundle $J^2(TX)$.*

For a theta characteristic ξ on X, the isomorphism $J^2(TX) \cong \mathrm{ad}(J^1(\xi^*))$ constructed in Proposition 4.1 using a projective structure P on X takes the connection on $J^2(TX)$ to the holomorphic connection on $\mathrm{ad}(J^1(\xi^*))$ defined by P.

Proof. A projective P on X gives a holomorphic connection D_ξ on $J^1(\xi^*)$, where ξ is a theta characteristic on X. The connection D_ξ induces a connection on $\mathrm{ad}(J^1(\xi^*))$, which, using the isomorphism in Proposition 4.1, gives a holomorphic connection on $J^2(TX)$. We need to show that this connection on $J^2(TX)$ does not depend on the choice of ξ.

If ξ_1 is another theta characteristic on X, then $\xi_1 \cong \xi \bigotimes \zeta$, where ζ is a holomorphic line bundle with $\zeta^{\otimes 2} = \mathcal{O}_X$. We noted earlier that ζ has a natural holomorphic connection and

$$J^1(\xi_1^*) = J^1(\xi^*) \otimes \zeta$$

(see (4.1)).

Consider the connection on $J^1(\xi^*) \bigotimes \zeta$ defined by the connection D_ξ on $J^1(\xi^*)$ and the natural connection on ζ. The connection D_{ξ_1} on $J^1(\xi_1^*)$ defined by the projective structure P is sent to this connection on $J^1(\xi^*) \bigotimes \zeta$ by the above isomorphism. From this it follows immediately that the connection on

$$\mathrm{ad}(J^1(\xi_1^*)) = \mathrm{ad}(J^1(\xi^*) \otimes \zeta) = \mathrm{ad}(J^1(\xi^*))$$

constructed using D_ξ coincides with the one constructed using D_{ξ_1}. In other words, the connection on $J^2(TX) \cong \mathrm{ad}(J^1(\xi^*))$ does not depend on the choice of ξ. This completes the proof of the proposition. □

The map constructed in Proposition 4.2 from the space of all projective structures on X to the space of all holomorphic connections on $J^2(TX)$ is injective. To prove this we recall that the projective structures on X are identified with the space of all holomorphic connections on the projective bundle $\mathbb{P}(J^1(\vartheta))$, where ϑ is a fixed line bundle with $\vartheta^{\otimes 2} \cong TX$. Therefore, the space of all projective structures on X is naturally embedded into

$$\mathcal{R} := \mathrm{Hom}(\pi_1(X), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}),$$

where the embedding sends a projective structure to the monodromy of the corresponding flat connection on $\mathbb{P}(J^1(\vartheta))$. Since the adjoint action of $\mathrm{PSL}(2,\mathbb{C})$ on its Lie algebra $\mathrm{sl}(2,\mathbb{C})$ is faithful, the map from the space of all projective structures on X to the space of all holomorphic connections on $\mathrm{ad}(J^1(\vartheta)) \cong J^2(TX)$ is injective.

Our aim in the rest of this section is to identify the connections on $J^2(TX)$ that arise from projective structures.

Let P be a projective structure on X and D the corresponding holomorphic connection on $J^2(TX)$ constructed in Proposition 4.2.

Note that $\bigwedge^3 J^2(TX) = \mathcal{O}_X$; it follows from (2.1). The connection on $\bigwedge^3 J^2(TX)$ induced by the connection D on $J^2(TX)$ is the trivial connection, that is, the induced connection has trivial monodromy. Indeed, this follows immediately from the fact that the connection D is induced by a $\mathrm{PSL}(2,\mathbb{C})$-connection on $\mathbb{P}(J^1(\vartheta))$ using the isomorphism in Proposition 4.1, where ϑ^* is a theta characteristic on X.

A holomorphic connection D_0 on $J^2(TX)$ gives an endomorphism of the vector bundle $J^2(TX)$. We will construct this endomorphism.

Take a point $x \in X$ and a vector $v \in J^2(TX)_x$ in the fiber over x. Let s_v be the (unique) flat section for the connection D_0 on $J^2(TX)$ defined around a connected simply connected open subset $U \subset X$ with $x \in U$ and satisfying the condition $s_v(x) = v$. Let $p(s_v)$ be the holomorphic section of TU, where

$$p : J^2(TX) \longrightarrow TX \tag{4.7}$$

is the composition $J^2(TX) \longrightarrow J^1(TX) \longrightarrow TX$ constructed using (2.1). Let

$$w \in J^2(TX)_x$$

be the vector defined by the section $p(s_v)$. Now we have a homomorphism of vector bundles

$$F_{D_0} : J^2(TX) \longrightarrow J^2(TX) \tag{4.8}$$

that sends any v to w constructed above from v.

The following lemma is straight-forward.

Lemma 4.3. *For the holomorphic connection D on $J^2(TX)$ arising from a projective structure P on X, we have $F_D = \mathrm{Id}_{J^2(TX)}$, where F_D is constructed in (4.8).*

Proof. Consider the isomorphism of vector bundles

$$J_{\mathbb{P}} : \mathbb{P}(W_0) \times H^0(\mathbb{P}(W_0), T\mathbb{P}(W_0)) \longrightarrow J^2(T\mathbb{P}(W_0)) \tag{4.9}$$

constructed earlier; see the construction of $I_{\mathbb{P}}$ in (4.3). We recall that $J_{\mathbb{P}}$ sends a vector field on $\mathbb{P}(W_0)$ to the restriction of it to the second order infinitesimal neighborhood of the points of $\mathbb{P}(W_0)$.

Note that the flat connection $J^2(T\mathbb{P}(W_0))$ for the standard (unique) projective structure on $\mathbb{P}(W_0)$ coincides with the one obtained from the trivialization of $J^2(T\mathbb{P}(W_0))$ defined by the isomorphism $J_{\mathbb{P}}$ in (4.9). In other words, flat sections of $J^2(T\mathbb{P}(W_0))$ are precisely the global vector fields on $\mathbb{P}(W_0)$ (by the isomorphism $J_{\mathbb{P}}$).

Consequently, the lemma is valid for the standard projective structure on $\mathbb{P}(W_0)$.

Since the connection D on $J^2(TX)$ is constructed by patching together the pull backs of the connection on $J^2(T\mathbb{P}(W_0))$ by holomorphic coordinate functions compatible with P, the fact that the lemma is valid for the standard projective structure on $\mathbb{P}(W_0)$ immediately implies that it is also valid for the projective structure P on X. This completes the proof of the lemma. □

The connection D is compatible with the Lie bracket operation of vector fields. This will be explained next.

As before, let D_0 be any holomorphic connection on $J^2(TX)$. Let

$$s, t \in H^0(U, J^2(TU)) \tag{4.10}$$

be flat sections on an open subset $U \subset X$ for the connection D_0. So

$$p(s), p(t) \in H^0(U, TU)$$

are holomorphic vector fields on U, where p is the projection in (4.7). So the Lie bracket $[s, t]$ is a holomorphic vector field on U. Let

$$\widehat{D}_0([s, t]) \in H^0(U, J^2(TU)) \tag{4.11}$$

be the section over U defined by the Lie bracket $[p(s), p(t)]$.

The following lemma is also straight-forward.

Lemma 4.4. *For the connection D on $J^2(TX)$ arising from a projective structure P on X, the section $\widehat{D}([s, t])$ constructed as in (4.11) using D is flat with respect to D, where s and t as in (4.10) are flat sections with respect to D.*

Proof. Since the Lie bracket of two globally defined holomorphic vector fields on $\mathbb{P}(W_0)$ is again a globally defined holomorphic vector field, and the isomorphism $J_{\mathbb{P}}$ in (4.9) takes the connection on $J^2(T\mathbb{P}(W_0))$ to the trivial connection on the trivial vector bundle $\mathbb{P}(W_0) \times H^0(\mathbb{P}(W_0))$ it follows that the lemma is valid for the connection on $J^2(T\mathbb{P}(W_0))$ arising from the standard (unique) projective structure on $\mathbb{P}(W_0)$. Now for the same reason given in the proof of Lemma 4.3, the fact that the lemma is valid for the projective structure on $\mathbb{P}(W_0)$ implies that it is valid for the projective structure P on X. This completes the proof of the lemma. □

Theorem 4.5. *Let D_0 be a holomorphic connection on $J^2(TX)$. This connection corresponds to a projective structure on X (by Proposition 4.2) if and only if the following three conditions hold:*

(1) *the connection on $\bigwedge^3 J^2(TX) = \mathcal{O}_X$ induced by D_0 coincides with the trivial connection on the trivial line bundles;*
(2) *the homomorphism F_{D_0} in (4.8) is the identity automorphism of $J^2(TX)$;*
(3) *for any two flat sections s, t as in (4.10) with respect to D_0, the section $\widehat{D}_0([s, t])$ in (4.11) is also flat with respect to D_0.*

Proof. We noted earlier that a connection on $J^2(TX)$ arising from a projective structure on X induces the trivial connection on the trivial line bundle $\bigwedge^3 J^2(TX)$. Combining this with Lemmas 4.3 and 4.4 we conclude that a connection on $J^2(TX)$ arising from a projective structure on X satisfies all the three conditions in the statement of the theorem.

Let D_0 be a holomorphic connection on $J^2(TX)$ satisfying the three conditions. We will construct a projective structure on X using D_0.

We first note that the third condition implies that the fibers of $J^2(TX)$ are equipped with a Lie algebra structure. To prove this, take any point $x \in X$ and two vectors $v, w \in J^2(TX)_x$ in the fiber over x. Let s (respectively, t) be the unique flat section of $J^2(TX)$ defined around x, for the connection D_0, such that $s(x) = v$ (respectively, $t(x) = w$). Sending the ordered pair v, w to $\widehat{D}_0([s, t])(x)$ we get a Lie algebra structure on the fiber $J^2(TX)_x$, where $\widehat{D}_0$ is as in (4.10).

The Lie algebra structure on $J^2(TX)_x$ will be denoted by $[-, -]$. We will now show that this three dimensional Lie algebra $J^2(TX)_x$ is isomorphic to sl(2, $\mathbb{C}$).

Let

$$0 \longrightarrow K_X \longrightarrow J^2(TX) \xrightarrow{q} J^1(TX) \longrightarrow 0 \tag{4.12}$$

be the exact sequence constructed in (2.1). If $(f_1(z) + z^3 g_1(z))\frac{\partial}{\partial z}$ and $(f_2(z) + z^3 g_2(z))\frac{\partial}{\partial z}$ are two vector fields defined around $0 \in \mathbb{C}$, where f_1, f_2 are polynomials of degree at most two, then the Lie bracket

$$\left[(f_1(z) + z^3 g_1(z))\frac{\partial}{\partial z}, (f_2(z) + z^3 g_2(z))\frac{\partial}{\partial z}\right] = \left[f_1(z)\frac{\partial}{\partial z}, f_2(z)\frac{\partial}{\partial z}\right] + z^2 g(z)\frac{\partial}{\partial z}$$

where g is a polynomial. Furthermore, given any polynomial $h(z)$ of degree at most one, we can find f_1 and f_2 as above (of degree at most two) with

$$\left[f_1(z)\frac{\partial}{\partial z}, f_2(z)\frac{\partial}{\partial z}\right] = h(z)\frac{\partial}{\partial z}.$$

Therefore, the second condition in the theorem implies that

$$q([J^2(TX)_x, J^2(TX)_x]) = J^1(TX)_x \tag{4.13}$$

for the Lie algebra structure defined on $J^2(TX)_x$ by D_0, where q is the projection in (4.12). The second condition implies that given any $\alpha \in J^2(TX)_x$, we can find a flat section s_α of $J^2(TX)$ defined around x such that the section $p(s_\alpha)$ restricts to α, where p is the projection in (4.7). Hence (4.13) is valid.

On the other hand, we have

$$\left[(z + z^3 g_1(z))\frac{\partial}{\partial z}, (z^2 + z^3 g_2(z))\frac{\partial}{\partial z}\right] = z^2\frac{\partial}{\partial z} + z^3 g_3(z)\frac{\partial}{\partial z}.$$

This, using the second condition in the theorem, immediately implies that

$$J^2(TX)_x \supset \text{kernel}(q(x)) \subset [J^2(TX)_x, J^2(TX)_x],$$

where $q(x)$ is the projection in (4.12).

The inclusion $\text{kernel}(q(x)) \subset [J^2(TX)_x, J^2(TX)_x]$ and (4.13) together imply that $J^2(TX)_x$ is spanned by the subset

$$[J^2(TX)_x, J^2(TX)_x] \subset J^2(TX)_x.$$

It is a straight-forward exercise to check that this implies that $J^2(TX)_x$ is isomorphic to sl(2, $\mathbb{C}$). Indeed, the facts that $[J^2(TX)_x, J^2(TX)_x]$ spans $J^2(TX)_x$ and $\dim J^2(TX)_x = 3$ together imply that the Lie algebra $J^2(TX)_x$ does not have any nonzero nilpotent ideal.

Take any point $x \in X$. Let $0 \neq v \in J^2(TX)_x$ be a nonzero nilpotent element of the Lie algebra satisfying the condition that

$$v \notin \text{kernel}(q(x)) \subset J^2(TX)_x,$$

where q is the projection in (4.12). To show that there exists such v, note that any nonzero nilpotent element in $\mathrm{sl}(2, \mathbb{C})$ is conjugate to the strictly upper triangular 2×2 matrix with 1 as the $(1, 2)$-th element. So the nilpotent elements constitute a one parameter family of lines in $J^2(TX)_x$. Therefore, there are nilpotent elements in the complement $J^2(TX)_x \setminus \mathrm{kernel}(q(x))$.

We will show that

$$0 \neq p(x)(v) \in T_x X \tag{4.14}$$

with p as in (4.7). For this note that for any $a, b \in \mathbb{C}$ we have

$$\left[(az + bz^2 + z^3 g_1(z))\frac{\partial}{\partial z}, (z^2 + z^3 g_2(z))\frac{\partial}{\partial z}\right] = az^2\frac{\partial}{\partial z} + z^3 g_3(z)\frac{\partial}{\partial z}.$$

In view of the second condition in the theorem, this immediately implies that for any

$$w \in \mathrm{kernel}(p(x))$$

(p is defined in (4.7)) the line

$$\mathrm{kernel}(q(x)) \subset [J^2(TX)_x, J^2(TX)_x]$$

(q is defined in (4.12)) is contained in an eigenspace for the adjoint action of w on $J^2(TX)_x$; the adjoint action on $\mathrm{kernel}(q(x))$ of the element in $\mathrm{kernel}(p(x))$ defined by $(az + bz^2 + z^3 g_1(z))\frac{\partial}{\partial z}$ (with respect to a holomorphic coordinate function z around x) is multiplication by a.

Consequently, no element in the complement

$$\mathrm{kernel}(p(x)) \setminus \mathrm{kernel}(q(x)) \subset J^2(TX)_x$$

is nilpotent. Indeed, each element in the above complement has a nonzero element in $\mathrm{kernel}(q(x))$ as an eigenvector for the adjoint action; on the other hand, the adjoint action of a nonzero nilpotent element $w \in \mathrm{sl}(2, \mathbb{C})$ has exactly one eigenspace, namely the line spanned by w.

Therefore, we conclude that (4.14) holds.

Let s_v be the (unique) flat section of $J^2(TX)$ defined around x (for the connection D_0) that satisfies the condition $s_v(x) = v$. Since $p(v) \neq 0$, there is a simply connected neighborhood U of x such that for all $y \in U$ we have $p(s_v(y)) \neq 0$. There is a unique holomorphic coordinate function

$$z : U \longrightarrow \mathbb{C} \tag{4.15}$$

such that

$$\frac{\partial}{\partial z} = p(s_v) \in H^0(U, TU) \tag{4.16}$$

with $z(x) = 0$ (we may need to shrink U to define the coordinate function).

We will construct a projective structure on X using the coordinate function defined as above around each point of X.

For this first note that there is a unique isomorphism of Lie algebras

$$F_v : J^2(TX)_x \longrightarrow \mathrm{sl}(2, \mathbb{C}) \tag{4.17}$$

such that

$$F_v(v) = A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.18}$$

and $F_v(\text{kernel}(q(x)))$ is the line in $\text{sl}(2, \mathbb{C})$ spanned by the transpose A^t, where A is defined in (4.18).

Now, take any $v' \in J^2(TX)_x$ that satisfies the conditions for v. In other words, v' is a nilpotent element of the Lie algebra $J^2(TX)_x$ with

$$0 \neq v' \notin \text{kernel}(q(x)).$$

There is a unique element

$$T \in \text{PSL}(2, \mathbb{C}) \tag{4.19}$$

which is the image of an element of the form

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \tag{4.20}$$

such that $\text{Ad}(T)(A) := TAT^{-1} = F_v(v')$, where F_v and A are defined in (4.17) and (4.18) respectively.

Define $s_{v'}$ exactly as s_v was defined. In other words, $s_{v'}$ is the (unique) flat section of $J^2(TX)$ defined around x (for the connection D_0) such that $s_{v'}(x) = v'$. Let

$$z' : U \longrightarrow \mathbb{C}$$

be the unique holomorphic coordinate function (as in (4.15)) around x with

$$\frac{\partial}{\partial z'} = p(s_{v'}) \in H^0(U, TU)$$

(as in (4.16)) with $z'(x) = 0$.

Now it is straight-forward to check that

$$z' = T \circ z,$$

where T is defined in (4.19). In other words, if T is the image of the matrix in (4.20), then $z' = az/(bz + c)$.

Since the coordinates z and z' differ by a Möbius transformation, we get a projective structure on any infinitesimal neighborhood of x. This projective structure is defined by z, and the projective structure does not depend on the choice of v.

To prove that this defines a projective structure on X, we need to show that for any $y \in Y$ and a neighborhood U_y on y equipped with the projective structure constructed as above using D_0, the two projective structures on $U \cap U_y$ coincide.

To prove this, consider the vector field $p(s_v)$ on U (see (4.16)). If instead of z we take another holomorphic coordinate function z_1 as in (4.15) satisfying the condition (4.16) but with $z_1(x) = c$ which need not be zero, then clearly,

$$z_1 = z + c.$$

Now $z \longrightarrow z+c$ is also a projective transformation. In other words, both the coordinate functions z_1 and z define the same projective structure on a neighborhood of x. Also, a composition of projective transformations is also a projective transformation. Consequently, the locally defined projective structures patch together compatibly to define a projective structure on X.

Let P_0 denote the projective structure on X constructed above from D_0. It is straight-forward to check that the connection on $J^2(TX)$ defined by P_0 using Proposition 4.2 coincides with D_0. Just note that these two connections coincide over the domain U in (4.15); both of these connections over U coincide with the connection on $J^2(T\mathbb{CP}^1)$ defined by the projective structure on $T\mathbb{CP}^1$.

If D_0 is defined by a projective structure P on X (using Proposition 4.2), then the projective structure constructed (as above) from D_0 clearly coincides with P. This completes the proof of the theorem. □

In the next section, using Theorem 4.5 we will give an alternative description of an orbifold projective structure.

5. Orbifold projective structure and connection

We now return to the general case where $\mathbb{D} = \sum_{i=1}^{\ell} \zeta_i$ need not be the empty set.

Let $J^2_{\mathbb{D}}(TX)$ denote the kernel of the projection

$$J^2(TX) \xrightarrow{p} TX \longrightarrow \frac{TX}{\text{image}(f_0)} = \bigoplus_{i=1}^{\ell} T_{\zeta_i}X, \tag{5.1}$$

where f_0 and p are as in (3.2) and (4.7) respectively. Therefore, we have an exact sequence of coherent sheaves

$$0 \longrightarrow J^2_{\mathbb{D}}(TX) \longrightarrow J^2(TX) \longrightarrow \bigoplus_{i=1}^{\ell} T_{\zeta_i}X \longrightarrow 0 \tag{5.2}$$

over X.

From (5.1) it follows that

$$\text{kernel}(p) \subset J^2_{\mathbb{D}}(TX). \tag{5.3}$$

For any $\zeta_i \in \mathbb{D}$, let

$$F^2_{\zeta_i} \subset (J^2_{\mathbb{D}}(TX))_{\zeta_i} \tag{5.4}$$

be the image of the fiber $(\text{kernel}(p))_{\zeta_i}$ by the inclusion homomorphism in (5.3). Now consider the line

$$(K_X)_{\zeta_i} = \text{kernel}(q(\zeta_i)) \subset J^2(TX)_{\zeta_i},$$

where q is the projection in (4.12). The image of $(K_X)_{\zeta_i}$ by the homomorphism in (5.3) defines a line

$$F^1_{\zeta_i} \subset F^2_{\zeta_i} \subset (J^2_{\mathbb{D}}(TX))_{\zeta_i} \tag{5.5}$$

with $F^2_{\zeta_i}$ defined in (5.4).

On the other hand, we have

$$J^2(TX) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\mathbb{D}) \subset J^2_{\mathbb{D}}(TX)$$

and the image of the fiber $(J^2(TX) \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(-\mathbb{D}))_{\zeta_i}$ is a line

$$G_{\zeta_i} \subset J^2_{\mathbb{D}}(TX)_{\zeta_i}. \tag{5.6}$$

It is easy to see that

$$J^2_{\mathbb{D}}(TX)_{\zeta_i} = F^2_{\zeta_i} \oplus G_{\zeta_i}, \tag{5.7}$$

where $F^2_{\zeta_i}$ and G_{ζ_i} are defined in (5.4) and (5.6) respectively.

Let ∇ be a logarithmic connection on the vector bundle $J^2_{\mathbb{D}}(TX)$ singular over the divisor $\mathbb{D}$. Let

$$\mathrm{Res}(\nabla, \zeta_i) \in \mathrm{End}(J^2_{\mathbb{D}}(TX)_{\zeta_i})$$

be the residue of ∇ over ζ_i.

Definition 5.1. We will say that a logarithmic connection ∇ on $J^2_{\mathbb{D}}(TX)$ singular over $\mathbb{D}$ *satisfies the residue condition* if for each $\zeta_i \in \mathbb{D}$ the following three conditions hold:

(1) the residue endomorphism $\mathrm{Res}(\nabla, \zeta_i)$ of the fiber $J^2_{\mathbb{D}}(TX)_{\zeta_i}$ preserves the decomposition in (5.7), and $\mathrm{Res}(\nabla, \zeta_i)$ acts on the line G_{ζ_i} as multiplication by $1/\varpi(\zeta_i)$ (the function ϖ is defined in (3.5));
(2) the endomorphism of $F^2_{\zeta_i}$ defined by $\mathrm{Res}(\nabla, \zeta_i)$ preserves the line $F^1_{\zeta_i}$ in (5.5), and it acts on $F^1_{\zeta_i}$ as multiplication by $(\varpi(\zeta_i) - 1)/\varpi(\zeta_i)$;
(3) the residue $\mathrm{Res}(\nabla, \zeta_i)$ induces the zero endomorphism of the quotient $F^2_{\zeta_i}/F^1_{\zeta_i}$.

Note that if ∇ satisfies the residue condition then $\mathrm{Res}(\nabla, \zeta_i)$ is semisimple, that is, $J^2_{\mathbb{D}}(TX)_{\zeta_i}$ is generated by the eigenvectors of $\mathrm{Res}(\nabla, \zeta_i)$.

Let ∇ be a logarithmic connection on $J^2_{\mathbb{D}}(TX)$ singular over $\mathbb{D}$. Take a point $x \in X' := X \setminus \mathbb{D}$ and take a vector $v \in J^2_{\mathbb{D}}(TX)_x$. Let s_v be the (unique) locally defined flat section, for the connection ∇, of $J^2_{\mathbb{D}}(TX)$ with $s_v(x) = v$ and defined on some connected open subset U containing x. Let $p_0(s_v)$ be the holomorphic section of TU, where

$$p_0 : J^2_{\mathbb{D}}(TX) \longrightarrow TX \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\mathbb{D}) \tag{5.8}$$

is the restriction to $J^2_{\mathbb{D}}(TX)$ of the projection p in (4.7). Let $w \in J^2(TX)_x$ be the vector defined by the vector field $p_0(s_v)$; note that on X' the two vector bundles $J^2_{\mathbb{D}}(TX)$ and $TX \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(-\mathbb{D})$ are identified with the vector bundles $J^2(TX)$ and TX respectively.

Now we have a homomorphism of vector bundles

$$F_\nabla : J^2_{\mathbb{D}}(TX) \longrightarrow J^2_{\mathbb{D}}(TX) \tag{5.9}$$

over X' that sends any v to w constructed above from v.

Similarly, as done in (4.11), given any two flat sections (for ∇)

$$s, t \in H^0(U, J^2_{\mathbb{D}}(TX))$$

defined over some open set $U \subset X'$, The Lie bracket $[p_0(s), p_0(t)]$ gives a section

$$\widehat{D}_\nabla([s, t]) \in H^0(U, J^2_{\mathbb{D}}(TX)), \tag{5.10}$$

where p_0 is the projection in (5.8).

Since $\bigwedge^3 J^2(TX) = \mathcal{O}_X$, using (5.2) it follows that

$$\bigwedge^3 J^2_{\mathbb{D}}(TX) = \mathcal{O}_X(-\mathbb{D}).$$

The line bundle $\mathcal{O}_X(-\mathbb{D})$ has a canonical logarithmic connection singular over $\mathbb{D}$. Indeed, the de Rham differential operator in (2.8) defines the connection operator

$$\mathrm{d} : \mathcal{O}_X(-\mathbb{D}) \longrightarrow K_X.$$

This connection on $\mathcal{O}_X(-\mathbb{D})$ is nonsingular over X' and its residue on each $\zeta_i \in \mathbb{D}$ is 1.

The following theorem follows from Theorem 4.5 and the use of a covering surface.

Theorem 5.2. *There is a natural bijective correspondence between the space of all orbifold projective structures on X and the space of all logarithmic connections ∇ on $J^2_{\mathbb{D}}(TX)$ singular over $\mathbb{D}$ satisfying the residue condition and also satisfying the following three conditions:*

(1) *the logarithmic connection on $\bigwedge^3 J^2_{\mathbb{D}}(TX) = \mathcal{O}_X(-\mathbb{D})$ induced by ∇ coincides with the canonical logarithmic connection on $\mathcal{O}_X(-\mathbb{D})$;*

(2) *the endomorphism*

$$F_\nabla : J^2_{\mathbb{D}}(TX)|_{X'} \longrightarrow J^2_{\mathbb{D}}(TX)|_{X'}$$

defined in (5.9) is the identity map;

(3) *the section*

$$\widehat{D}_\nabla([s,t]) \in H^0(U, J^2_{\mathbb{D}}(TX))$$

in (5.10) is flat with respect to ∇ for any flat sections $s, t \in H^0(U, J^2_{\mathbb{D}}(TX))$ with $U \subset X'$.

Proof. Consider the Galois covering $\gamma : Y \longrightarrow X$ constructed in (3.7). For any $i \in [1, \ell]$, let

$$y_i := (\gamma^{-1}(\zeta_i))_{\mathrm{red}} \subset Y$$

be the set-theoretic inverse image. Set

$$W := \gamma^* J^2_{\mathbb{D}}(TX) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\left(\sum_{i=1}^{\ell}(\varpi(\zeta_i) - 1)y_i\right) \tag{5.11}$$

to be the vector bundle over Y, where ϖ is the function in (3.5).

For any $i \in [1, \ell]$, consider the quotient space

$$Q_i := \frac{J^2_{\mathbb{D}}(TX)_{\zeta_i}}{F^1_{\zeta_i} \oplus G_{\zeta_i}},$$

where $F^1_{\zeta_i}$ and G_{ζ_i} are defined in (5.5) and (5.6) respectively. So Q_i is a quotient of the sheaf $J^2_{\mathbb{D}}(TX)$ supported on the reduced point ζ_i. Therefore,

$$Q'_i := \gamma^{-1}(Q_i) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y\left(\sum_{i=1}^{\ell}(\varpi(\zeta_i) - 1)y_i\right)$$

is a quotient of the sheaf W (defined in (5.11)) supported over the nonreduced divisor $\gamma^{-1}(\zeta_i) = \varpi(\zeta_i)y_i$ of Y. This quotient map will be denoted by g_i.

Let $\overline{Q}'_i$ denote the restriction of the sheaf Q'_i to the subscheme

$$(\varpi(\zeta_i) - 1)y_i \subset \varpi(\zeta_i)y_i.$$

So we have a natural projection

$$f_i : Q'_i \longrightarrow \overline{Q}'_i. \tag{5.12}$$

Let

$$\mathcal{E}_1 \subset W \tag{5.13}$$

be the kernel of the composition

$$W \xrightarrow{\sum_{i=1}^{\ell} g_i} \bigoplus_{i=1}^{\ell} Q'_i \xrightarrow{\bigoplus_{i=1}^{\ell} f_i} \bigoplus_{i=1}^{\ell} \overline{Q}'_i$$

(recall that $g_i : W \longrightarrow Q'_i$ is the quotient map), where f_i is defined in (5.12).

Now, for any $i \in [1, \ell]$, consider the quotient space

$$G_i = \frac{J^2_{\mathbb{D}}(TX)_{\zeta_i}}{F^2_{\zeta_i}},$$

where $F^2_{\zeta_i}$ is defined in (5.4). So G_i is a quotient sheaf of $J^2_{\mathbb{D}}(TX)$ supported on the reduced point ζ_i. Therefore, as before,

$$G'_i := \gamma^{-1}(G_i) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \left(\sum_{i=1}^{\ell} (\varpi(\zeta_i) - 1) y_i \right)$$

is a quotient of W (defined in (5.11)) supported on $\varpi(\zeta_i) y_i$. This quotient map will be denoted by g'_i.

Let $\overline{G}'_i$ denote the restriction of the sheaf G'_i to the subscheme

$$(\varpi(\zeta_i) - 1) y_i \subset \varpi(\zeta_i) y_i.$$

So we have a natural projection

$$f'_i : G'_i \longrightarrow \overline{G}'_i. \tag{5.14}$$

Let

$$\mathcal{E}_2 \subset W \tag{5.15}$$

be the kernel of the composition

$$W \xrightarrow{\sum_{i=1}^{\ell} g'_i} \bigoplus_{i=1}^{\ell} G'_i \xrightarrow{\bigoplus_{i=1}^{\ell} f'_i} \bigoplus_{i=1}^{\ell} \overline{G}'_i$$

(recall that $g'_i : W \longrightarrow G'_i$ is the quotient map), where f'_i is defined in (5.14).

Finally, let

$$\mathcal{E} \subset \mathcal{E}_1 \cap \mathcal{E}_2 \subset W \tag{5.16}$$

be the intersection, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in (5.13) and (5.15) respectively.

It may be mentioned at this point that the reason behind the above construction of $\mathcal{E}$ is to ensure that for any logarithmic connection ∇ on $J^2_{\mathbb{D}}(TX)$ satisfying the residue condition and also satisfying the three conditions in the theorem, the logarithmic connection $\gamma^*\nabla$ on $\gamma^* J^2_{\mathbb{D}}(TX)$ defines a nonsingular connection on $\mathcal{E}$. This will be explained in detail later.

Note that over the complement $Y \setminus \gamma^{-1}(\mathbb{D})$ all the vector bundles W, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}$ are naturally identified with

$$\gamma^* J^2(TX)|_{Y\setminus\gamma^{-1}(\mathbb{D})} = \gamma^* J^2_{\mathbb{D}}(TX)|_{Y\setminus\gamma^{-1}(\mathbb{D})} = J^2(TY)|_{Y\setminus\gamma^{-1}(\mathbb{D})}.$$

Any automorphism of Y lifts naturally to $J^2(TY)$. So, the action of the Galois group G on Y lifts to an action of G on $J^2(TY)$ as vector bundle automorphisms. The vector bundle $\gamma^* J^2_{\mathbb{D}}(TX)$ being a pullback is equipped with a lift of the action of G as vector bundle automorphisms.

We need the following lemma.

Lemma 5.3. *The holomorphic sections of $J^2(TY)$ invariant under the action of G on $J^2(TY)$, that are defined on open subsets invariant under the action of G on Y, are identified with the holomorphic sections of $\gamma^* J^2_{\mathbb{D}}(TX)$ invariant under the action of G on $\gamma^* J^2_{\mathbb{D}}(TX)$ that are defined on G-invariant open subsets of Y.*

For the vector bundle $\mathcal{E}$ constructed in (5.16), the identification

$$\mathcal{E}|_{Y\setminus\gamma^{-1}(\mathbb{D})} = J^2(TY)|_{Y\setminus\gamma^{-1}(\mathbb{D})}$$

extends to an isomorphism of $\mathcal{E}$ with $J^2(TY)$ over Y.

Proof. To prove the first part of the lemma we note that the coherent sheaf on X that associates to any open subset $U \subset X$ the space of all G-invariant holomorphic 1-forms on $\gamma^{-1}(U)$ is identified with the sheaf of holomorphic 1-forms on X (see [4, page 88, Lemma 3.7]). Similarly, the coherent sheaf on X that associates to any open subset $U \subset X$ the space of all G-invariant holomorphic vector fields on $\gamma^{-1}(U)$ is identified with the sheaf defined by the line bundle $TX \bigotimes \mathcal{O}_X(-\mathbb{D})$ over X.

To prove the above assertion, set $U_1 \subset \mathbb{C}$ to be the unit disk, and consider the map

$$U_1 \longrightarrow U_1 \tag{5.17}$$

defined by $z \longmapsto z^n$. The vector field $z\frac{\partial}{\partial z}$ on U_1 is left invariant under the action of the Galois group $\mathbb{Z}/n\mathbb{Z}$, and furthermore, all the invariant holomorphic vector fields on U_1 are generated by this one as module over invariant holomorphic functions. On the other hand, if $w = z^n$, then $z\frac{\partial}{\partial z} = nw\frac{\partial}{\partial w}$.

Therefore, $TX \bigotimes \mathcal{O}_X(-\mathbb{D})$ coincides with the sheaf defined by (locally defined) G-invariant vector fields on Y. The coherent sheaf on X that associates to any open subset $U \subset X$ the space of all G-invariant holomorphic functions on $\gamma^{-1}(U)$ is identified with the sheaf defined by the trivial line bundle on X. (See the proof of Lemma 3.3 for a similar argument.)

The action of G on $J^2(TY)$ clearly preserves the filtration

$$K_Y \subset \text{kernel}(p_Y) \subset J^2(TY),$$

where $p_Y : J^2(TY) \longrightarrow TY$ is the natural projection (defined exactly as in (4.7)) and K_Y is the kernel of the projection $J^2(TY) \longrightarrow J^1(TY)$. The action of G on K_Y coincides with the given by the natural lift of automorphisms. The quotient $\text{kernel}(p_Y)/K_Y$ is the trivial line bundle over Y equipped with the trivial lift of the action of G on Y, that is, the group G acts diagonally on $Y \times \mathbb{C}$ with the action of G on $\mathbb{C}$ being the trivial one. The quotient $J^2(TY)/\text{kernel}(p_Y)$ is TX with the natural lift of the action of G to TX.

Using these observations it follows that the sheaf on X that associates to any open subset $U \subset X$ the space of all G-invariant holomorphic sections of $J^2(T\gamma^{-1}(U))$ is identified with the sheaf defined by the vector bundle $J^2_{\mathbb{D}}(TX)$. This proves the first part of the lemma.

To prove the second part of the lemma we need the following observation. Consider the action of Galois group $\mathbb{Z}/n\mathbb{Z}$ for the map in (5.17) on the trivial line bundle $U_1 \times \mathbb{C}$ with U_1 as in (5.17), defined as follows: the action of the generator $1 \in \mathbb{Z}/n\mathbb{Z}$ sends any $(z, c) \in U_1 \times \mathbb{C}$ to $(\exp(2\pi\sqrt{-1}/n)z, \exp(2\pi\sqrt{-1}k/n)c)$, where k is a fixed integer in $[0, n-1]$. The invariant sections of the trivial line bundle for this action are generated by the section defined by the function $z \longmapsto z^k$. In particular, the order of vanishing at zero of the generating section is strictly less than n.

The above observation and the first part of the lemma together imply that

$$\gamma^* J^2_{\mathbb{D}}(TX) \subset J^2(TY) \subset \gamma^* J^2_{\mathbb{D}}(TX) \otimes \mathcal{O}_Y\left(\sum_{i=1}^{\ell}(\varpi(\zeta_i) - 1)y_i\right). \tag{5.18}$$

In other words, $J^2(TY)$ is a subsheaf of W defined in (5.11).

The following decomposition into a direct sum of line bundles

$$J^2(TU_1) = \frac{\partial}{\partial z} \otimes_{\mathbb{C}} \mathcal{O}_{U_1} \oplus z\frac{\partial}{\partial z} \otimes_{\mathbb{C}} \mathcal{O}_{U_1} \oplus z^2\frac{\partial}{\partial z} \otimes_{\mathbb{C}} \mathcal{O}_{U_1}$$

over the unit disk U_1, where z is the standard coordinate on U_1, is left invariant by the action of $\mathbb{Z}/n\mathbb{Z}$ considered above (the Galois group for the map $z \longrightarrow z^n$).

Now using the earlier observation that invariant sections are generated by the function $z \longmapsto z^k$ it follows that the construction of $\mathcal{E}$ from W coincides with the construction of the subsheaf $J^2(TY) \subset W$ in (5.18). This completes the proof of the lemma. □

Remark 5.4. If a finite group Γ acts on a complex projective manifold Y_1, and if E is a holomorphic vector bundle over Y_1 equipped with a lift of the action of Γ as vector bundle automorphisms, then in [5] a construction is given to recover E from the invariant sheaf $(f_*E)^\Gamma$ and some data over $(f_*E)^\Gamma$ called *parabolic structure*, where f is the projection of Y to Y/Γ (the construction of [5] works under some assumptions on the ramification divisor and the restriction of E over it). The above construction of $\mathcal{E}$ from $J^2_{\mathbb{D}}(TX)$ is a special case of the construction of [5].

Continuing with the proof of the theorem, let P be an orbifold projective structure on X. As we saw in the proof of Lemma 3.2, the orbifold projective structure P defines a projective structure $\mathcal{P}$ on the covering Riemann surface Y in (3.7) which is left invariant by the action of the Galois group G on Y.

Using Theorem 4.5, $\mathcal{P}$ gives a holomorphic connection ∇' on $J^2(TY)$ which is left invariant by the action of G on $J^2(TY)$. Using the isomorphism $\mathcal{E} \cong J^2(TY)$ in Lemma 5.3, this connection ∇' defines a connection ∇'' on $\mathcal{E}$.

Since the divisor $y_i \subset Y$ is left invariant by the action of the Galois group G for the covering map γ, the line bundle $\mathcal{O}_Y(y_i)$ is equipped with a canonical lift of the action of G on Y. Therefore, the line bundle $\mathcal{O}_Y(\sum_{i=1}^{\ell} m_i y_i)$, where $m_i \in \mathbb{Z}$, is equipped with a lift of the action of G. The vector bundle $\gamma^* J^2_{\mathbb{D}}(TX)$ being a pullback is also equipped with a lift of the action of G. Therefore, W defined in (5.11) is equipped with a lift of the action of G on Y. From the definition of $\mathcal{E}_1$ (respectively, $\mathcal{E}_2$) in (5.13) (respectively, (5.15)) it follows immediately that the action of G on W leaves the subsheaf $\mathcal{E}_1$ (respectively, $\mathcal{E}_2$) invariant. In other words, both $\mathcal{E}_1$ and $\mathcal{E}_2$ have an induced action of G. Therefore, the action of G on W leaves $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$ in (5.16) invariant.

The isomorphism in Lemma 5.3 takes the induced action of G on $\mathcal{E}$ to the action of G on $J^2(TY)$. Therefore, the induced action of G on $\mathcal{E}$ leaves the connection ∇'' invariant.

We will recall a property of a logarithmic connection which will be used.

Let V be a holomorphic vector bundle over X and $F_0 \subset V_{x_0}$ a subspace of the fiber over $x_0 \in X$. Let V' be the vector bundle defined by the exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V_{x_0}/F_0 \longrightarrow 0. \tag{5.19}$$

Let ∇^0 be a logarithmic connection on V singular over the point x_0. Then ∇^0 induces a logarithmic connection on V' if and only if the residue

$$\mathrm{Res}(\nabla^0, x_0) \in \mathrm{End}(V_{x_0})$$

leaves the subspace $F_0 \subset V_{x_0}$ invariant. Assume that $\mathrm{Res}(\nabla^0, x_0)$ preserves subspace $F_0 \subset V_{x_0}$. Let R_0 (respectively, R_1) be the endomorphism of F_0 (respectively, V_{x_0}/F_0) induced by $\mathrm{Res}(\nabla^0, x_0)$. The kernel of the homomorphism

$$f_{x_0} : V'_{x_0} \longrightarrow F_0 \subset V_{x_0} \tag{5.20}$$

of fibers (obtained by restricting the exact sequence (5.19) to x_0) is identified with $(V_{x_0}/F_0) \bigotimes \ell_0$, where ℓ_0 is the fiber over x_0 of the line bundle $\mathcal{O}_X(-x_0)$. If ∇^1 is the logarithmic connection on V' induced by ∇^0, then there is an isomorphism

$$T : F_0 \oplus (V_{x_0}/F_0) \longrightarrow V'_{x_0}$$

such that

(1) $T(V_{x_0}/F_0) = (V_{x_0}/F_0) \bigotimes \ell_0 = \mathrm{kernel}(f_{x_0})$ (the homomorphism f_{x_0} is defined in (5.20)) and $T(w) = w \otimes w_0$, where $w \in V_{x_0}/F_0$ and w_0 is a fixed element in ℓ_0 independent of w;
(2) the homomorphism $F_0 \longrightarrow F_0$ induced by T is the identity map (the first condition implies that T induces a homomorphism of quotients, and $V'_{x_0}/\mathrm{kernel}(f_{x_0}) = F_0$);
(3) $T \circ (R_1 + \mathrm{Id}_{V_{x_0}/F_0}) = \mathrm{Res}(\nabla^1, x_0) \circ T$ on V_{x_0}/F_0 (this condition implies that $\mathrm{Res}(\nabla^1, x_0)$ preserves $\mathrm{kernel}(f_{x_0})$, and hence $\mathrm{Res}(\nabla^1, x_0)$ induces an endomorphism of the quotient F_0);
(4) $R_0 = R'$ on F_0, where $R' \in \mathrm{End}(F_0)$ is the endomorphism induced by $\mathrm{Res}(\nabla^1, x_0)$ (see (3)).

Using the above criterion it follows that the connection ∇'' on $\mathcal{E}$ induces a logarithmic connection $\widehat{\nabla}$ on $\gamma^* J^2_{\mathbb{D}}(TX)$. Since ∇'' is left invariant by the action of G on $\mathcal{E}$ we conclude that the natural action of G on $\gamma^* J^2_{\mathbb{D}}(TX)$ leaves the logarithmic connection $\widehat{\nabla}$ invariant.

Therefore, $\widehat{\nabla}$ descends to a logarithmic connection ∇ on $J^2_{\mathbb{D}}(TX)$. From the above property of the residue of the induced connection it follows that ∇ satisfy the residue condition (see Definition 5.1 for residue condition). Furthermore, from the properties of the connection ∇' on $J^2(TY)$ described in Theorem 4.5 it follows immediately that the logarithmic connection ∇ on $J^2_{\mathbb{D}}(TX)$ satisfies all the three conditions in the statement of the theorem.

For the converse direction, let ∇ be a logarithmic connection on $J^2_{\mathbb{D}}(TX)$ satisfying the conditions in the statement of the theorem. Let $\gamma^*\nabla$ be the pulled back logarithmic connection on the vector bundle $\gamma^* J^2_{\mathbb{D}}(TX)$ over Y.

Consider the kernel $F' := \mathrm{kernel}(f_{x_0}) \subset V'_{x_0}$ of the homomorphism in (5.20). Recall that the quotient V'_{x_0}/F' is identified with $F_0 \subset V_{x_0}$. Let ∇_0 be a logarithmic connection on V' singular over the point x_0. The connection ∇_0 induces a logarithmic connection on the vector bundle V in (5.19) if and only if the residue

$$\mathrm{Res}(\nabla_0, x_0) \in \mathrm{End}(V'_{x_0})$$

leaves the subspace $F' \subset V'_{x_0}$ invariant. Assume that $\mathrm{Res}(\nabla_0, x_0)$ preserves the subspace F'. Let R_0 (respectively, R_1) be the endomorphism of F' (respectively, $F_0 = V'_{x_0}/F'$) induced by $\mathrm{Res}(\nabla_0, x_0)$. If ∇_1 is the logarithmic connection on V induced by ∇_0, then there is an isomorphism

$$T : F' \oplus F_0 \longrightarrow V_{x_0}$$

such that

(1) $T(w) = w$ for any $w \in F_0$;
(2) $T(w \otimes w_0) = w$ for all $w \in V_{x_0}/F_0$, where w_0 is a fixed nonzero element of the line ℓ_0 (so w_0 is independent of w, recall that $F' = (V_{x_0}/F_0) \bigotimes \ell_0$);
(3) $T \circ (R_0 - \mathrm{Id}_{F'}) = \mathrm{Res}(\nabla_1, x_0) \circ T$ on F';
(4) $R_1 = R'$ on F_0, where $R' \in \mathrm{End}(F_0)$ is induced by $\mathrm{Res}(\nabla_1, x_0)$.

Using these properties of a logarithmic connection together with the given hypothesis that ∇ satisfies the residue condition it follows that the logarithmic connection $\gamma^*\nabla$ on $\gamma^* J^2_{\mathbb{D}}(TX)$ induces a regular holomorphic connection on the vector bundle $\mathcal{E}$ constructed in (5.16). (To show that a logarithmic connection is actually a regular connection, it suffices to show that the residue at each singular point is zero.)

Let ∇' be the regular connection on $\mathcal{E} \cong J^2(TY)$ induced by $\gamma^*\nabla$ (see Lemma 5.3 for the isomorphism). The connection ∇' is evidently left invariant by the action of the Galois group G on $J^2(TY)$. Since ∇ satisfies the three conditions in the statement of the theorem it follows immediately that the connection ∇' on $J^2(TY)$ satisfies the three conditions in Theorem 4.5. Therefore, using Theorem 4.5 the connection ∇' gives a G-invariant projective structure on Y. This projective structure, being G-invariant, descends to an orbifold projective structure on X.

The two constructions, namely from logarithmic connections to orbifold projective structures and vice versa, are inverses of each other. This completes the proof of the theorem. □

6. Differential operator associated to orbifold projective structures

In the first part of this final section we will assume that $\mathbb{D}$ is the zero divisor (= empty set).

6.1. *The case of* $\mathbb{D} = 0$

Let W_0 be a complex vector space of dimension two. In (4.3) and (4.9) we saw that

$$J^2(T\mathbb{P}(W_0)) \cong \mathbb{P}(W_0) \times H^0(\mathbb{P}(W_0), T\mathbb{P}(W_0)) \cong \mathbb{P}(W_0) \times \mathrm{sl}(W_0)$$

with the isomorphism defined by restricting global vector fields to the second order infinitesimal neighborhood of points of $\mathbb{P}(W_0)$. Therefore, we have splitting of the exact sequence

$$0 \longrightarrow K^{\otimes 2}_{\mathbb{P}(W_0)} \longrightarrow J^3(T\mathbb{P}(W_0)) \longrightarrow J^2(T\mathbb{P}(W_0)) \longrightarrow 0 \qquad (6.1)$$

in (2.1) that sends a global vector field to the third order infinitesimal neighborhood of points of $\mathbb{P}(W_0)$. More precisely, for any $x \in \mathbb{P}(W_0)$ and $v \in J^2(T\mathbb{P}(W_0))_x$, the homomorphism

$$J^2(T\mathbb{P}(W_0))_x \longrightarrow J^3(T\mathbb{P}(W_0))_x$$

giving the splitting of (6.1) sends v to the element in $J^3(T\mathbb{P}(W_0))_x$ obtained by restricting the vector field on $\mathbb{P}(W_0)$ corresponding to v to the third order infinitesimal neighborhood of x. The splitting of (6.1) gives a homomorphism

$$J^3(T\mathbb{P}(W_0)) \longrightarrow K^{\otimes 2}_{\mathbb{P}(W_0)}.$$

This homomorphism defines a differential operator

$$D_0 \in H^0(\mathbb{P}(W_0), \mathrm{Diff}^3_{\mathbb{P}(W_0)}(T\mathbb{P}(W_0), K^{\otimes 2}_{\mathbb{P}(W_0)})) \tag{6.2}$$

whose symbol is the constant function 1 (see (2.2)).

The local system on $\mathbb{P}(W_0)$ defined by the sheaf of solutions of D_0 (defined in (6.2)) is identified with the local system defined by the flat connection on $J^2(T\mathbb{P}(W_0))$ given by its trivialization in (4.9).

Let

$$z : \mathbb{P}(W_0) \longrightarrow \mathbb{C} \cup \{\infty\}$$

be any globally defined holomorphic coordinate function on $\mathbb{P}(W_0)$. So on $z^{-1}(\mathbb{C})$ any holomorphic vector field is of the form $f(z)\frac{\partial}{\partial z}$, where f is an entire function. The differential operator D_0 in (6.2) satisfies the identity

$$D_0\left(f(z)\frac{\partial}{\partial z}\right) = \frac{\partial^3 f}{\partial z^3}(\mathrm{d}z)^{\otimes 2}. \tag{6.3}$$

It is easy to check that if D_0 is of the above form with respect to some locally defined holomorphic coordinate function z on $\mathbb{P}(W_0)$, then z is the restriction of a globally defined holomorphic coordinate function of the above type.

Now let Y be a compact connected Riemann surface equipped with a projective structure P.

Since the differential operator D_0 in (6.2) is equivariant under the actions of $\mathrm{GL}(W_0)$ on $T\mathbb{P}(W_0)$ and $K^{\otimes 2}_{\mathbb{P}(W_0)}$, it induces a differential operator

$$D_Y \in H^0(Y, \mathrm{Diff}^3_Y(TY, K^{\otimes 2}_Y)) \tag{6.4}$$

in the following way.

Take any holomorphic coordinate function

$$\psi : Y \subset U \longrightarrow \mathbb{P}(W_0)$$

compatible with the projective structure P. The differential $\mathrm{d}\psi$ identifies TU (respectively, $K^{\otimes 2}_U$) with $\psi^* T\mathbb{P}(W_0)$ (respectively, $\psi^* K^{\otimes 2}_{\mathbb{P}(W_0)}$). So the differential operator D_0 gives a holomorphic differential operator

$$D_U \in H^0(U, \mathrm{Diff}^3_U(TU, K^{\otimes 2}_U))$$

over the open subset $U \subset Y$. Since D_0 intertwines the actions of $\mathrm{GL}(W_0)$ on $T\mathbb{P}(W_0)$ and $K^{\otimes 2}_{\mathbb{P}(W_0)}$, these locally defined differential operators D_U patch together compatibly to define a globally defined differential operator D_Y as in (6.4).

Since the symbol of D_0 is the constant function 1, it follows immediately that the symbol of D_Y is also the constant function 1.

Since the local system defined by the sheaf of solution of D_0 is identified with the local system defined by the natural connection on $J^2(T\mathbb{P}(W_0))$, it follows that the local system on Y defined by the sheaf of solution of the differential operator D_Y is identified with the local system defined by the flat connection on $J^2(TY)$ constructed in Proposition 4.2 from P. Indeed, this is an immediate consequence of the fact that the connection on $J^2(TY)$ in Proposition 4.2 is constructed by patching together the connections on $J^2(TU)$ given by the natural connection on

$J^2(T\mathbb{P}(W_0))$ using a coordinate function over U compatible with the given projective structure P.

The operator D_Y determines the projective structure P. To reconstruct P from D_Y, take holomorphic coordinate functions on Y such that D_Y is of the form (6.3) in terms of the coordinate functions. Such coordinate functions are compatible with P, and hence P is reconstructed using these coordinate functions.

6.2. *The general case of* $\mathbb{D}$

Now we remove the assumption that $\mathbb{D} = 0$.

Let $\mathcal{P}$ be an orbifold projective structure on X. Take $\gamma : Y \longrightarrow X$ as in (3.7). The projective structure $\mathcal{P}$ on X gives a projective structure P on Y which is left invariant by the action of the Galois group G on Y (see the proof of Lemma 3.2).

So P gives a G-invariant differential operator D_Y as in (6.4). Such a differential operator D_Y descends to a differential operator

$$D_X \in H^0(X, \mathrm{Diff}^3_X(TX \otimes \mathcal{O}_X(-\mathbb{D}), K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D}))). \tag{6.5}$$

Indeed, this follows immediately from the fact that the sheaf on X defined by the G-invariant local sections of TY (respectively, $K_Y^{\otimes 2}$) is identified with the sheaf defined by $TX \otimes \mathcal{O}_X(-\mathbb{D})$ (respectively, $K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})$); see the first two paragraphs in the proof of Lemma 5.3 as well as the proof of the isomorphism in (3.8). Note that since D_Y is left invariant by the action of G, if s is a locally defined G-invariant holomorphic section of TY, then $D_Y(s)$ is a G-invariant locally defined holomorphic section of $K_Y^{\otimes 2}$.

The symbol of D_X (defined in (6.5)) is a section of $\mathcal{O}_X(2\mathbb{D})$; see the definition of symbol in (2.3). Since the symbol of D_Y is the constant function 1, it follows that the symbol of D_X is

$$1 \in H^0(X, \mathcal{O}_X) \subset H^0(X, \mathcal{O}_X(2\mathbb{D}))$$

(the section defined by the constant function 1).

The orbifold projective structure $\mathcal{P}$ is determined by the differential operator D_X. To reconstruct $\mathcal{P}$ from D_X note that D_X determines D_Y. Therefore, the projective structure P on Y is determined by D_X. Hence $\mathcal{P}$ is determined by D_X.

The flat connection on $X' = X \setminus \mathbb{D}$ corresponding to the local system on X' defined by the sheaf of solutions of the differential operator D_X extends to a logarithmic connection on the vector bundle $J^2_{\mathbb{D}}(TX)$ defined in (5.2). To prove this we first recall that the local system on Y defined by the sheaf of solutions sheaf of D_Y corresponds to the flat connection on $J^2(TY)$ for the projective structure P; the connection was constructed in Proposition 4.2. Since D_Y descends to X as D_X, and the connection on $J^2(TY)$ descends to the logarithmic connection on $J^2_{\mathbb{D}}(TX)$ defined by $\mathcal{P}$ (see the proof of Theorem 5.2), we conclude that the logarithmic connection on $J^2_{\mathbb{D}}(TX)$ constructed from $\mathcal{P}$ (constructed in Theorem 5.2) is an extension of the connection over X' defined by the sheaf of solutions of D_X.

Therefore, we have the following variation of Theorem 5.2.

Theorem 6.1. *There is a natural bijective correspondence between the space of all orbifold projective structures on X and the subset of*

$$H^0(X, \mathrm{Diff}^3_X(TX \otimes \mathcal{O}_X(-\mathbb{D}), K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})))$$

consisting of all differential operator D_X *such that*

(1) *the symbol of* D_X *is the section of* $H^0(X, \mathcal{O}_X(2\mathbb{D}))$ *given by the constant function* 1 *,*
(2) *the flat connection on* $X \setminus \mathbb{D}$ *corresponding to the sheaf of solutions of* D_X *extends as a logarithmic connection on* $J^2_{\mathbb{D}}(TX)$ *over* X*, and*
(3) *this logarithmic connection on* $J^2_{\mathbb{D}}(TX)$ *satisfies all the conditions in Theorem 5.2.*

6.3. Kernel of the differential operator

We will describe the section

$$\mathcal{K}^{-1}(D_X) \in H^0(4\Delta, p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta)) \tag{6.6}$$

corresponding to the differential operator D_X (constructed in (6.5)) by the isomorphism $\mathcal{K}$ in (2.6).

Identify $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$ by sending any $c \in \mathbb{C}$ to the line in $\mathbb{C}^2$ defined by $(1, c)$. This identification gives a meromorphic coordinate function on $\mathbb{CP}^1$ which will be denoted by z.

Take any holomorphic coordinate function

$$\phi_i : V_i \longrightarrow U_i \tag{6.7}$$

as in (3.6) compatible with the given orbifold projective structure $\mathcal{P}$ on X. Let $\Delta_{V_i} \subset V_i \times V_i$ be the reduced diagonal divisor and

$$q_{i,j} : V_i \times V_i \longrightarrow V_i,$$

$j = 1, 2$, the projection to the j-th factor. Over $V_i \times V_i$ consider the meromorphic form

$$\omega_i := \frac{(\mathrm{d}z_1)^{\otimes 2} \otimes (\mathrm{d}z_2)^{\otimes 2}}{(z_1 - z_2)^4} \in H^0(V_i \times V_i, q_{i,1}^* K_{V_i}^{\otimes 2} \otimes q_{i,2}^* K_{V_i}^{\otimes 2} \otimes \mathcal{O}_{V_i\times V_i}(4\Delta_{V_i})),$$

where (z_1, z_2) is the holomorphic coordinate function on $V_i \times V_i$ defined by $z_j(v_1, v_2) = z(v_j)$, $j = 1, 2$.

Restricting this section to $n\Delta_{V_i}$, $n \geq 1$, we get a section

$$\omega_{i,n} \in H^0(n\Delta_{V_i}, (q_{i,1}^* K_{V_i}^{\otimes 2} \otimes q_{i,2}^* K_{V_i}^{\otimes 2} \otimes \mathcal{O}_{V_i\times V_i}(4\Delta_{V_i}))|_{n\Delta_{V_i}}).$$

Let $\Delta_{U_i} \subset U_i \times U_i$ be the diagonal. Using the covering map ϕ_i the section ω_i on the infinitesimal neighborhoods on Δ_{V_i} descends to section

$$\widehat{\omega}_i \in H^0(4\Delta_{U_i}, (p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta))|_{4\Delta_{U_i}})$$

(actually, we get sections over each $n\Delta_{U_i}$, but here we are interested only in the section over $4\Delta_{U_i}$).

To construct $\widehat{\omega}_i$ first note that the section

$$\frac{(\mathrm{d}z_1)^{\otimes 2} \otimes (\mathrm{d}z_2)^{\otimes 2}}{(z_1 - z_2)^4} \in H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, (K_{\mathbb{CP}^1}^{\otimes 2} \boxtimes K_{\mathbb{CP}^1}^{\otimes 2}) \otimes \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(4\Delta_{\mathbb{CP}^1})) \tag{6.8}$$

is invariant under the diagonal action of the Möbius group $\mathrm{PSL}(2, \mathbb{C})$. In particular, the earlier defined section ω_i is invariant under the diagonal action of the Galois group for the covering map ϕ_i in (6.7). The diagonal $\Delta_{V_i} \subset V_i \times V_i$ is left invariant by the diagonal action of the Galois group and the quotient is Δ_{U_i}. Also, we saw that the sheaf on X defined by the G-invariant local sections of $K_Y^{\otimes 2}$ is identified with the sheaf defined by $K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})$ (see the proof of the isomorphism in (3.8)). Therefore, ω_i descends to a section $\widehat{\omega}_i$, over $4\Delta_{U_i}$, of the above type.

Since the section in (6.8) is invariant under the diagonal action of $\mathrm{PSL}(2,\mathbb{C})$, it follows immediately that for another holomorphic coordinate function $\phi_j : V_j \longrightarrow U_j$ as in (6.7) compatible $\mathcal{P}$, the two sections $\widehat{\omega}_i$ and $\widehat{\omega}_j$ coincide over $(4\Delta_{U_i}) \cap (4\Delta_{U_i}) \subset X \times X$.

Consequently, these locally defined sections $\widehat{\omega}_i$ patch together compatibly to define a holomorphic section of the line bundle

$$\mathcal{L} := p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta) \tag{6.9}$$

over $4\Delta \subset X \times X$.

Let $S_{\mathcal{P}}$ denote this section of $\mathcal{L}$ over 4Δ constructed from the projective structure $\mathcal{P}$. We will show that $S_{\mathcal{P}}$ coincides with the section $\mathcal{K}^{-1}(D_X)$ in (6.6).

To prove this, first consider the section $\frac{(\mathrm{d}z_1)^{\otimes 2}\otimes(\mathrm{d}z_2)^{\otimes 2}}{(z_1-z_2)^4}$ in (6.8) over $\mathbb{CP}^1 \times \mathbb{CP}^1$. The restriction of this section to $4\Delta_{\mathbb{CP}^1}$ is actually the kernel of the differential operator D_0 over $\mathbb{CP}^1$ constructed in (6.2); here $\Delta_{\mathbb{CP}^1}$ is the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Indeed, this is an immediate consequence of the local expression of D_0 given in (6.3). Since D_X is constructed from D_0 using coordinates compatible with the given projective structure $\mathcal{P}$, we conclude that the section $\mathcal{K}^{-1}(D_X)$ in (6.6) coincides with $S_{\mathcal{P}}$.

We will list properties of the section $\mathcal{K}^{-1}(D_X)$ defined in (6.6).

Using the Poincaré adjunction formula, the restriction to Δ of $\mathcal{L}$ (defined in (6.9)) is identified with the line bundle $\mathcal{O}_X(2\mathbb{D})$ after identifying Δ with X. Since the symbol of the differential operator D_X is $1 \in H^0(X, \mathcal{O}_X(2\mathbb{D}))$ (see Theorem 6.1), from the description of symbol given in (2.7) it follows immediately that the restriction of $\mathcal{K}^{-1}(D_X)$ to Δ is given by $1 \in H^0(X, \mathcal{O}_X(2\mathbb{D}))$.

Let

$$\tau_X : X \times X \longrightarrow X \times X$$

be the involution defined by $(x, y) \longmapsto (y, x)$. The pullback $\tau_X^*\mathcal{L}$ is canonically identified with $\mathcal{L}$ (defined in (6.9)). Indeed, this is an immediate consequence of the fact that τ_X leaves the diagonal Δ invariant. In other words, the involution τ_X lifts naturally to $\mathcal{L}$.

The section $S_{\mathcal{P}}$ of $\mathcal{L}$ defined over 4Δ is left invariant by τ_X (the involution leaves 4Δ invariant). This follows immediately from the fact that the section in (6.8) over $\mathbb{CP}^1 \times \mathbb{CP}^1$ is left invariant by the involution of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Let $U \subset X \times X$ be an analytic open subset with $\tau_X(U) = U$, and let f be a holomorphic function defined over U such that $f = f \circ \tau_X$. Then the order of vanishing of f on the divisor $\Delta \cap U \subset U$ must be even. Consequently, a τ_X-invariant holomorphic section over U of the line bundle $\mathcal{L}$ (defined in (6.9)) vanishing of order at least $2k - 1$ over $\Delta \cap U$ must vanish of order at least $2k$ over $\Delta \cap U$, where k is an integer.

Therefore, if s and s' are two sections of $\mathcal{L}$ over 4Δ such that

(1) both s and s' are left invariant by the action of τ_X on the line bundle,
(2) $s|_{3\Delta} = s'|_{3\Delta}$,

then $s = s'$. In other words, any τ_X invariant section of $\mathcal{L}$ over 3Δ extends uniquely to a τ_X invariant section of $\mathcal{L}$ over 4Δ. So the section $S_{\mathcal{P}}$ over 4Δ is determined by its restriction to 3Δ.

Take any point $\zeta_i \in \mathbb{D}$. Let

$$\iota : X \longrightarrow X \times X$$

be the inclusion defined by $x \longmapsto (x, \zeta_i)$. We have

$$\iota_X^* \mathcal{L} = K_X^{\otimes 2} \otimes \mathcal{O}_X(5\mathbb{D}) \otimes \xi_0, \tag{6.10}$$

where ξ_0 is the trivial line bundle over X with fiber $(K_X^{\otimes 2} \bigotimes \mathcal{O}_X(\mathbb{D}))_{\zeta_i}$.

Let $s \in H^0(3\Delta, \mathcal{L})$ be a holomorphic section over 3Δ which is invariant under the involution τ_X. Set

$$\beta_s := \iota^*(s) \in H^0(3\zeta_i, \iota^*\mathcal{L}) \tag{6.11}$$

over the nonreduced divisor $3\zeta_i \subset X$, where ι is defined above.

Note that if we used the embedding $x \longrightarrow (\zeta_i, x)$ instead of ι, then the fact that both $\mathcal{L}$ and s are invariant under the involution τ_X implies that the section in (6.11) remains unchanged.

Assume that β_s in (6.11) vanishes at ζ_i of order two. In view of this assumption, using (6.10) it follows that β_s is a section

$$\beta_s \in H^0(3\zeta_i, K_X^{\otimes 2} \otimes \mathcal{O}_X(3\mathbb{D}) \otimes \xi_0).$$

Since $\mathcal{O}_X(\mathbb{D})_{\zeta_i} = T_{\zeta_i} X$ (the Poincaré adjunction formula), the fiber over ζ_i of the line bundle $K_X^{\otimes 2} \otimes \mathcal{O}_X(3\mathbb{D}) \otimes \xi_0$ is identified with $\mathbb{C}$. Hence we have

$$\beta_s|_{\zeta_i} \in \mathbb{C}. \tag{6.12}$$

Now set s to be the section $S_{\mathcal{P}}|_{3\Delta}$, where $S_{\mathcal{P}}$ is the holomorphic section of $\mathcal{L}$ over 4Δ constructed using the projective structure $\mathcal{P}$. Since $S_{\mathcal{P}}$ is invariant under the involution τ_X, and the symbol of the differential operator D_X vanishes at ζ_i of order two (see Theorem 6.1), we conclude that $S_{\mathcal{P}}$ satisfies all the above conditions on s.

Using the fact that the logarithmic connection on $J^2_{\mathbb{D}}(TX)$ defined by the sheaf of solutions of the operator D_X in (6.5) satisfies the residue condition (see Definition 5.1, Theorems 5.2 and 6.1) it follows that $\beta_{S(\mathcal{P})}|_{\zeta_i}$ in (6.12) is $1/\varpi(\zeta_i)$. The eigenvalue of the eigenvector G_{ζ_i} in Definition 5.1(1) coincides with $\beta_{S(\mathcal{P})}|_{\zeta_i}$.

Combining the above observations we have the following theorem.

Theorem 6.2. *There is a natural bijective correspondence between the space of all orbifold projective structures on X and the space of all sections*

$$s \in H^0(3\Delta, p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta))$$

over $3\Delta \subset X \times X$ satisfying the following conditions

(1) *the section s is invariant under the involution of $X \times X$,*
(2) *the restriction of s to Δ coincides with the section of $H^0(X, \mathcal{O}_X(2\mathbb{D}))$ given by the constant function 1 (after identifying Δ with X),*
(3) $\beta_s|_{\zeta_i} \in \mathbb{C}$ *in (6.12) is* $1/\varpi(\zeta_i)$.

Let

$$s, s' \in H^0(3\Delta, p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta))$$

be two sections satisfying only the first two of the three conditions in Theorem 6.2. Therefore, $s - s'$ vanishes on Δ and it is invariant under the involution of $X \times X$. As we saw earlier, these imply that $s - s'$ actually vanishes on 2Δ. Therefore,

$$s - s' \in H^0(X, \mathcal{O}_X(2\mathbb{D}) \otimes K_X^{\otimes 2})$$

after identifying Δ with X (and using the Poincaré adjunction formula). Using the Poincaré adjunction formula the fiber of $\mathcal{O}_X(2\mathbb{D}) \bigotimes K_X^{\otimes 2}$ over any $\zeta_i \in \mathbb{D}$ is identified with $\mathbb{C}$. It is easy to see that

$$(s - s')(\zeta_i) \in \mathbb{C}$$

coincides with $\beta_s|_{\zeta_i} - \beta_{s'}|_{\zeta_i} \in \mathbb{C}$, with β as in (6.12). Therefore, if s and s' also satisfy the third condition in Theorem 6.2, then the section $s - s'$ of $\mathcal{O}_X(2\mathbb{D}) \bigotimes K_X^{\otimes 2}$ vanishes on $\mathbb{D}$. In that case, we have

$$s - s' \in H^0(X, \mathcal{O}_X(\mathbb{D}) \otimes K_X^{\otimes 2}).$$

Conversely, any $\alpha \in H^0(X, \mathcal{O}_X(\mathbb{D}) \otimes K_X^{\otimes 2})$ gives a section

$$\alpha' \in H^0(3\Delta, p_1^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes p_2^*(K_X^{\otimes 2} \otimes \mathcal{O}_X(\mathbb{D})) \otimes \mathcal{O}_{X\times X}(4\Delta)).$$

If $S_{\mathcal{P}}$ is a section as in Theorem 6.2 corresponding to an orbifold projective structure $\mathcal{P}$ on X, then $s + \alpha'$ defines an orbifold projective structure using Theorem 6.2. This way, the space of all orbifold projective structures on X is an affine space for the vector space $H^0(X,\ \mathcal{O}_X(\mathbb{D}) \bigotimes K_X^{\otimes 2})$. This is a reformulation of Lemma 3.3.

References

[1] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957) 181–207.
[2] A. Beilinson, V. Drinfeld, Opers (preprint), math.AG/0501398.
[3] D. Ben-Zvi, I. Biswas, Theta functions and Szegö kernels, Int. Math. Res. Not. (24) (2003) 1305–1340.
[4] I. Biswas, Green–Lazarsfeld sets and the logarithmic Dolbeault complex for Higgs line bundles, Int. Math. Res. Not. (2) (1996) 83–91.
[5] I. Biswas, Parabolic bundles as orbifold bundles, Duke Math. J. 88 (1997) 305–325.
[6] I. Biswas, P.A. Gastesi, S. Govindarajan, Parabolic Higgs bundles and Teichmüller space for punctured surfaces, Trans. Amer. Math. Soc. 349 (1997) 1551–1560.
[7] I. Biswas, A.K. Raina, Projective structures on a Riemann surface, II, Int. Math. Res. Not. (13) (1999) 685–716.
[8] P. Deligne, Equations Différentielles à Points Singuliers Réguliers, in: Lecture Notes in Math., no. 163, Springer-Verlag, Berlin, 1970.
[9] R.C. Gunning, Lectures on Riemann surfaces, in: Mathematical Notes, vol. 2, Princeton University Press, Princeton, New Jersey, 1966.
[10] N.J. Hitchin, The self-duality equation on a Riemann surface, Proc. London Math. Soc. 55 (1987) 59–126.
[11] R. Mandelbaum, Branched structures on Riemann surfaces, Trans. Amer. Math. Soc. 163 (1972) 261–275.
[12] M. Namba, Branched coverings and algebraic functions, in: Pitman Research Notes in Mathematical Series, vol. 161, Scientific & Technical, Harlow, John Wiley & Sons, Inc., New York, 1987.
[13] B. Nasatyr, B. Steer, Orbifold Riemann surfaces and the Yang-Mills-Higgs equations, Ann. Scuola Norm. Sup. Pisa 22 (1995) 595–643.